IRMA Lectures in Mathematics and Theoretical Physics 15

Edited by Christian Kassel and Vladimir G. Turaev

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Renormalization and Galois Theories

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2000 Mathematical Subject Classification (primary; secondary): 14F42, 34M30, 34M35, 34M37, 34M40, 81T15, 81T18, 81T75; 11S40, 33B30, 34M50

ISBN 978-3-03719-073-9

Key words: Renormalization, motives, noncommutative geometry, Hopf algebras, differential Galois groups, Borel summation, Gevrey asymptotics, Stokes phenomenon, resurgent functions

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

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CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Typeset using the author's TEX files: I. Zimmermann, Freiburg
Printed in Germany

9 8 7 6 5 4 3 2 1

Preface

This volume presents a selection of worked-out lectures, delivered by mathematicians and theoretical physicists, that were held in March 2006 at the workshop "Renormalization and Galois theories" at CIRM, Marseilles, France. The contributions include the courses given by Caterina Consani, Vincent Rivasseau, and David Sauzin, together with articles written by Yves André, Kurusch Ebrahimi-Fard and Dominique Manchon, Michael Hoffman, Frédéric Menous, and Stefan Weinzierl.

August 2009

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Foreword

The CIRM meeting revolved around some of the interactions of four very active topics with origin in pure mathematics and in physics:

- Renormalization in Quantum Field Theory
- Differential Galois theory
- Noncommutative geometry
- Motives and Galois theory

The articles that form this volume all illustrate the richness of these interactions. The recent years have witnessed the emergence of a much better mathematical understanding of perturbative renormalization both in its Galois aspects related to the ambiguity inherent in the renormalization group and the role of the Birkhoff decomposition, as well as in the deep arithmetic nature of the numbers that appear as residues of Feynman graphs in the renormalization process. The Birkhoff decomposition plays a crucial role, together with the Hopf algebra of Feynman graphs, in the conceptual understanding of the perturbative renormalization process. This result of our collaboration with D. Kreimer is recalled, and largely extended to regularization processes more general than dimensional regularization, in the article of Kurusch Ebrahimi-Fard and Dominique Manchon.

There is a striking similarity between the ambiguity group occurring in physics, and its underpinning as the "cosmic Galois group" in joint work with M. Marcolli, and the ambiguities that occur in the resummation processes of divergent series (Stokes phenomenon, resurgence, etc.). Frederic Menous has developed a mechanism of Birkhoff decomposition to treat conjugacy problems that arise in the study of solutions of differential equations (analytic or formal). For some equations, the obstacles in the formal conjugacy are reflected in the fact that the associated characters of a Hopf algebra (here a shuffle Hopf algebra) appear to be ill-defined. The analogy with the need for a renormalization scheme (dimensional regularization, Birkhoff decomposition) in Quantum Field Theory becomes obvious for such equations and delivers a wide range of toy models.

The article of David Sauzin is a careful account of the mould calculus of Écalle, a powerful combinatorial tool which yields surprisingly explicit formulas for the series attached to an analytic germ of singular vector field or of map. His article for this volume illustrates the method for the case of the saddle-node of a two-dimensional vector field.

The work of Kreimer and Broadhurst opened up a new area of interactions between the highly involved computations of Quantum Field Theory, where numbers such as $\zeta(3)$ naturally occur as ingredients of computations of physical quantities, and deep aspects of arithmetic, such as the study of periods, in the theory of motives.

Stefan Weinzierl reviews in his contribution the connections between Feynman integrals and multiple polylogarithms. He gives a thorough introduction to loop integrals, the Mellin–Barnes transformation, shuffle algebras and multiple polylogarithms. Finally, he discusses how certain Feynman integrals evaluate to multiple polylogarithms.

In his article, Michael Hoffman copes with various Hopf algebras which have their origin in renormalization; he shows in particular how combinatorial Dyson–Schwinger equations can profitably be solved by lifting them from a commutative algebra to a non-commutative one.

The aim of the enticing text of Yves André is to indicate what Grothendieck's theory of motives has to say, at least conjecturally, on the question of an analogue of Galois theory for transcendental numbers, and to promote the idea that periods should have well-defined conjugates and a Galois group that permutes them transitively.

The article of Caterina Consani starts by an enlightening historical survey of the theory of pure motives in algebraic geometry and then reviews some of the recent developments of this theory in noncommutative geometry, namely the new theory of endomotives, which was developed in joint work with C. Consani and M. Marcolli. It shows how a natural extension of the simplest category of motives, namely the Artin motives, to the larger category of algebraic endomotives quickly yields, by using cyclic cohomology and the thermodynamics of noncommutative spaces, the spectral realization of the zeros of the Riemann zeta function.

Finally, the article of Vincent Rivasseau and Fabien Vignes-Tourneret is a review of the fast growing subject of renormalization of Quantum Field Theory on noncommutative spaces. The Grosse–Wulkenhaar breakthrough opened up a new era by realizing that the right propagator in noncommutative field theory is not the ordinary commutative propagator, but it has to be modified to obey Langmann–Szabo duality. Grosse and Wulkenhaar were able to compute the corresponding propagator in the so-called "matrix base" which transforms the Moyal product into a matrix product, and to use this representation to prove perturbative renormalisability. V. Rivasseau, in joint work with R. Gurau, J. Magnen and F. Vignes-Tourneret, using direct space methods, provided recently a new proof that the Grosse–Wulkenhaar scalar Φ^4-theory on the Moyal space $\mathbb{R}^4$ is renormalisable to all orders in perturbation theory. Their review is remarkably clear and accessible.

In all of the mentioned subjects there is now a flurry of activity and the present book is a perfect introduction to these very lively topics of research, relevant both in physics and pure mathematics.

Alain Connes

Contents

Noncommutative geometry and motives
(à quoi servent les endomotifs?)

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Abstract. This paper gives a short and historical survey on the theory of pure motives in algebraic geometry and reviews some of the recent developments of this theory in noncommutative geometry. The second part of the paper outlines the new theory of endomotives and some of its relevant applications in number theory.

1 Introduction

This paper is based on three lectures I gave at the Conference on "Renormalization and Galois theories" that was held in Luminy, at the Centre International de Rencontres Mathématiques (CIRM), in March 2006. The purpose of these talks was to give an elementary overview on classical motives (pure motives) and to survey on some of the recent developments of this theory in noncommutative geometry, especially following the introduction of the notion of an *endomotive*.

It is likely to expect that the reader acquainted with the literature on motives theory will not fail to notice the allusion, in this title, to the paper [18] in which P. Deligne states that in spite of the lack of essential progress on the problem of constructing "relevant" algebraic cycles, the techniques supplied by the theory of motives remain a powerful tool in algebraic geometry and arithmetic.

The assertion on the lack of relevant progress on algebraic cycles seems, unfortunately, still to apply at the present time, fifteen years after Deligne wrote his paper. Despite the general failure of testing the Standard Conjectures, it is also true that in these recent years the knowledge on motives has been substantially improved by several new results and also by some unexpected developments.

Motives were introduced by A. Grothendieck with the aim to supply an intrinsic explanation for the analogies occurring among various cohomological theories in algebraic geometry. They are expected to play the role of a universal cohomological

[*]Work partially supported by NSF-FRG grant DMS-0652431. The author wishes to thank the organizers of the Conference for their kind invitation to speak and the CIRM in Luminy-Marseille for the pleasant atmosphere and their support.

theory by also furnishing a linearization of the theory of algebraic varieties and in the original understanding they were expected to provide the correct framework for a successful approach to the Weil's Conjectures on the zeta-function of a variety over a finite field.

Even though the Weil's Conjectures have been proved by Deligne without appealing to the theory of motives, an enlarged and in part still conjectural theory of mixed motives has in the meanwhile proved its usefulness in explaining conceptually some intriguing phenomena arising in several areas of pure mathematics, such as Hodge theory, K-theory, algebraic cycles, polylogarithms, L-functions, Galois representations etc.

Very recently, some new developments of the theory of motives to number theory and quantum field theory have been found or are about to be developed, with the support of techniques supplied by noncommutative geometry and the theory of operator algebras.

In number theory, a conceptual understanding of the main result of [9] on the interpretation proposed by A. Connes of the Weil explicit formulae as a Lefschetz trace formula over the noncommutative space of adèle classes, requires the introduction of a generalized category of motives inclusive of spaces which are highly singular from a classical viewpoint.

The problem of finding a suitable enlargement of the category of (smooth projective) algebraic varieties is combined with the even more compelling one of the definition of a generalized notion of correspondences. Several questions arise already when one considers special types of zero-dimensional noncommutative spaces, such as the space underlying the quantum statistical dynamical system defined by J. B. Bost and Connes in [6] (the BC-system). This space is a simplified version of the adèles class space of [9] and it encodes in its group of symmetries, the arithmetic of the maximal abelian extension of $\mathbb{Q}$.

In this paper I give an overview on the theory of endomotives (algebraic and analytic). This theory has been originally developed in the joint paper [10] with A. Connes and M. Marcolli and has been applied already in our subsequent work [12]. The category of endomotives is the minimal one that makes it possible to understand conceptually the role played by the absolute Galois group in several dynamical systems that have been recently introduced in noncommutative geometry as generalizations of the BC-system, which was our motivating and prototype example.

The category of endomotives is a natural enlargement of the category of Artin motives: the objects are noncommutative spaces defined by semigroup actions on projective limits of Artin motives. The morphisms generalize the notion of algebraic correspondences and are defined by means of étale groupoids to account for the presence of the semigroup actions.

Endomotives carry a natural Galois action which is inherited from the Artin motives and they have both an algebraic and an analytic description. The latter is particularly useful as it provides the data of a quantum statistical dynamical system, via the implementation of a *canonical time evolution* (a one-parameter family of automorphisms) which is associated by the theory of M. Tomita (*cf.* [37]) to an initial state (probability measure) assigned to an analytic endomotive. This is the crucial new development supplied by the theory of operator-algebras to a noncommutative C^*-algebra and in particular to the algebra of the BC-system.

The implication in number theory is striking: the time evolution implements on the dual system a scaling action which combines with the action of the Galois group to determine on the cyclic homology of a suitable noncommutative motive associated to the original endomotive, a characteristic zero analog of the action of the Weil group on the étale cohomology of an algebraic variety. When these techniques are applied to the endomotive of the BC-system or to the endomotive of the adèles class space, the main implication is the spectral realization of the zeroes of the corresponding L-functions.

These results supply a first answer to the question I raised in the title of this paper (à quoi servent les endomotifs?). An open and interesting problem is connected to the definition of a higher dimensional theory of noncommutative motives and in particular the introduction of a theory of noncommutative elliptic motives and modular forms. A related problem is of course connected to the definition of a higher dimensional theory of geometric correspondences. The comparison between algebraic correspondences for motives and geometric correspondences for noncommutative spaces is particularly easy in the zero-dimensional case, because the equivalence relations play no role. In noncommutative geometry, algebraic cycles are naturally replaced by bi-modules, or by classes in equivariant KK-theory. Naturally, the original problem of finding "interesting" cycles pops up again in this topological framework: a satisfactory solution to this question seems to be one of the main steps to undertake for a further development of these ideas.

2 Classical motives: an overview

The theory of motives in algebraic geometry was established by A. Grothendieck in the 1960s: 1963–69 (*cf.* [36], [22]). The foundations are documented in the unpublished manuscript [20] and were discussed in a seminar at the Institut des Hautes Études Scientifiques, in 1967. This theory was conceived as a fundamental machine to develop Grothendieck's "long-run program" focused on the theme of the connections between geometry and arithmetic.

At the heart of the philosophy of motives sit Grothendieck's speculations on the existence of a universal cohomological theory for algebraic varieties defined over a base field k and taking values into an abelian, tensor category. The study of this problem originated as the consequence of a general dissatisfaction connected to an extensive use of topological methods in algebraic geometry, with the result of producing several but insufficiently related cohomological theories (Betti, de Rham, étale, etc.). The typical example is furnished by a family of homomorphisms $H^i_{\text{et}}(X, \mathbb{Q}_\ell) \to H^i_{\text{et}}(Y, \mathbb{Q}_\ell)$ connecting the groups of étale cohomology of two (smooth, projective) varieties, as the prime number ℓ varies, which are not connected, in general, by any sort of (canonical) relation.

The definition of a contravariant functor (functor of motivic cohomology)

$$h\colon \mathcal{V}_k \to \mathcal{M}_k(\mathcal{V}_k), \quad X \mapsto h(X)$$

from the category $\mathcal{V}_k$ of projective, smooth, irreducible algebraic varieties over k to a semi-simple abelian category of pure motives $\mathcal{M}_k(\mathcal{V}_k)$ is also tied up with the definition of a universal cohomological theory through which every other classical, cohomology $H^\bullet$ (here understood as contravariant functor) should factor by means of the introduction of a fiber functor (realization $\otimes$-functor) ω connecting $\mathcal{M}_k(\mathcal{V}_k)$ to the abelian category of (graded) vector-spaces over $\mathbb{Q}$

$$\begin{tikzcd} \mathcal{V}_k \arrow[r, "H^\bullet"] \arrow[dr, "h"'] & \mathrm{grVect}_{\mathbb{Q}} \\ & \mathcal{M}_k(\mathcal{V}_k). \arrow[u, "\omega"'] \end{tikzcd}$$

Following Grothendieck's original viewpoint, the functor h should implement the sought for mechanism of compatibilities (in étale cohomology) and at the same time it should also describe a universal linearization of the theory of algebraic varieties.

The definition of the category $\mathcal{M}_k(\mathcal{V}_k)$ arose from a classical construction in algebraic geometry which is based on the idea of extending the collection of algebraic morphisms in $\mathcal{V}_k$ by including the (algebraic) *correspondences*. A correspondence between two objects X and Y in $\mathcal{V}_k$ is a multi-valued map which connects them. An algebraic correspondence is defined by means of an algebraic cycle in the cartesian product $X \times Y$. The concept of (algebraic) correspondence in geometry is much older than that of a motive: it is in fact already present in several works of the Italian school in algebraic geometry (*cf.* Severi's theory of correspondences on algebraic curves).

Grothendieck's new intuition was that the whole philosophy of motives is regulated by the theory of (algebraic) correspondences:

"*...J'appelle motif sur k quelque chose comme un groupe de cohomologie ℓ-adique d'un schema algébrique sur k, mais considérée comme indépendant de ℓ, et avec sa structure entière, ou disons pour l'instant sur $\mathbb{Q}$, déduite de la théorie des cycles algébriques...*" (*cf.* [16], Lettre 16.8.1964).

Motives were envisioned with the hope to explain the intrinsic relations between integrals of algebraic functions in one or more complex variables. Their ultimate goal was to supply a machine that would guarantee a generalization of the main results of Galois theory to systems of polynomials equations in several variables. Here, we refer in particular to a higher-dimensional analog of the well-known result which describes the linearization of the Galois–Grothendieck correspondence for the category $\mathcal{V}_k^o$ of étale, finite k-schemes

$$\mathcal{V}_k^o \xrightarrow{\sim} \{\text{finite sets with } \mathrm{Gal}(\bar{k}/k)\text{-action}\}, \quad X \mapsto X(\bar{k})$$

by means of the equivalence between the category of Artin motives and that of the representations of the absolute Galois group.

In the following section we shall describe how the fundamental notions of the theory of motives arose from the study of several classical problems in geometry and arithmetic.

2.1 A first approach to motives

A classical problem in algebraic geometry is that of computing the solutions of a finite set of polynomial equations

$$f_1(X_1,\dots,X_m)=0,\ \dots,\ f_r(X_1,\dots,X_m)=0$$

with coefficients in a finite field $\mathbb{F}_q$. This study is naturally formalized by introducing the generating series

$$\zeta(X,t)=\exp\left(\sum_{m\geq 1}\nu_m\frac{t^m}{m}\right) \tag{2.1}$$

which is associated to the algebraic variety $X=V(f_1,\dots,f_m)$ that is defined as the set of the common zeroes of the polynomials $f_1,\dots,f_r$.

Under the assumption that X is smooth and projective, the series (2.1) encodes the complete information on the number of the rational points of the algebraic variety, through the coefficients $\nu_m=|X(\mathbb{F}_{q^m})|$. The integers ν_m supply the cardinality of the set of the rational points of X, computed in successive finite field extensions $\mathbb{F}_{q^m}$ of the base field $\mathbb{F}_q$.

Intersection theory furnishes a general way to determine the number ν_m as intersection number of two algebraic cycles on the cartesian product $X\times X$; namely the diagonal Δ_X and the graph Γ_{Fr^m} of the m-th iterated composite of the Frobenius morphism on the scheme $(X,\mathcal{O}_X)$:

$$\mathrm{Fr}\colon X\to X;\quad \mathrm{Fr}(P)=P,\quad f(\underline{x})\mapsto f(\underline{x}^q)\quad \text{for all } f\in\mathcal{O}_X(U),\ U\subset X \text{ open.}$$

The Frobenius endomorphism is in fact an interesting example of *correspondence*, perhaps the most interesting one, for algebraic varieties defined over finite fields. As a correspondence it induces a commutative diagram

$$\begin{tikzcd} H^*(X\times X) \arrow[r, "-\cap\Gamma_{\mathrm{Fr}}"] & H^*(X\times X) \arrow[d, "(p_2)_*"] \\ H^*(X) \arrow[u, "p_1^*"] \arrow[r, "\mathrm{Fr}^*"'] & H^*(X) \end{tikzcd} \tag{2.2}$$

in étale cohomology. Through the commutativity of the above diagram one gets a way to express the action of the induced homomorphism in cohomology, by means of the formula

$$\mathrm{Fr}^*(c)=(p_2)_*(p_1^*(c)\cap\Gamma_{\mathrm{Fr}})\quad \text{for all } c\in H^*(X),$$

where $p_i : X \times X \to X$ denote the two projection maps. Here, 'algebraic' refers to the algebraic cycle $\Gamma_{\mathrm{Fr}} \subset X \times X$ that performs such a correspondence.

For particularly simple algebraic varieties, such as projective spaces $\boldsymbol{P}^n$, the computation of the integers ν_m can be done by applying an elementary combinatorial argument based on the set-theoretical description of the space $\boldsymbol{P}^n(k) = k^{n+1} \setminus \{0\}/k^\times$ (k = any field). This has the effect to produce the interesting description

$$|\boldsymbol{P}^n(\mathbb{F}_{q^m})| = \frac{q^{m(n+1)} - 1}{q^m - 1} = 1 + q^m + q^{2m} + \cdots + q^{mn}. \tag{2.3}$$

This decomposition of the set of the rational points of a projective space was certainly a first source of inspiration in the process of formalizing the foundations of the theory of motives. In fact, one is naturally led to wonder on the casuality of the decomposition (2.3), possibly ascribing such a result to the presence of a cellular decomposition on the projective space which induces a related break-up on the set of the rational points. Remarkably, A. Weil proved that a similar formula holds also in the more general case of a smooth, projective algebraic curve $C_{/\mathbb{F}_q}$ of genus $g \geq 0$. In this case one shows that

$$|C(\mathbb{F}_{q^m})| = 1 - \sum_{i=1}^{2g} \omega_i^m + q^m; \quad \omega_i \in \bar{\mathbb{Q}},\ |\omega_i| = q^{1/2}. \tag{2.4}$$

These results suggest that (2.3) and (2.4) are the manifestation of a deep and intrinsic structure that governs the geometry of algebraic varieties.

The development of the theory of motives has in fact shown to us that this structure reveals itself in several contexts: topologically, manifests its presence in the decomposition of the cohomology $H^*(X) = \bigoplus_{i \geq 0} H^i(X)$, whereas arithmetically it turns out that it is the same structure that controls the decomposition of the series (2.1) as a rational function of t:

$$\zeta(X, t) = \frac{\prod_{i \geq 0} \det(1 - t\mathrm{Fr}^* | H^{2i+1}_{\mathrm{et}}(X))}{\prod_{i \geq 0} \det(1 - t\mathrm{Fr}^* | H^{2i}_{\mathrm{et}}(X))}.$$

This is in fact a consequence of the description of the integers ν_m supplied by the Lefschetz–Grothendieck trace formula (*cf.* [23])

$$|X(\mathbb{F}_{q^m})| = \sum_{i \geq 0} (-1)^i \operatorname{tr}((\mathrm{Fr}^m)^* | H^i_{\mathrm{et}}(X)).$$

2.2 Grothendieck's pure motives

Originally, Grothendieck proposed a general framework for a so called category of *numerically effective motives* $\mathbf{M}(k)_{\mathbb{Q}}$ over a field k and with rational coefficients. This category is defined by enlarging the category $\mathcal{V}_k$ of smooth, projective algebraic varieties over k (and algebraic morphisms) by following the so-called procedure of pseudo-abelian envelope. This construction is performed in two steps: at first one enlarges the set of morphisms of $\mathcal{V}_k$ by including (rational) algebraic correspondences of degree zero, modulo numerical equivalence, then one performs a pseudo-abelian envelope by formally including among the objects, kernels of idempotent morphisms.

Let us assume for simplicity that the algebraic varieties are irreducible (the general case is then deduced from this, by additivity). For any given $X, Y \in \mathrm{Obj}(\mathcal{V}_k)$, one works with correspondences $f : X \dashrightarrow Y$ which are elements of codimension equal to $\dim X$ in the rational graded algebra

$$A^*(X \times Y) = C^*(X \times Y) \otimes \mathbb{Q}/ \sim_{\text{num}}$$

of algebraic cycles modulo *numerical equivalence*. We recall that two algebraic cycles on an algebraic variety X are said to be numerically equivalent $Z \sim_{\text{num}} W$, if

$$\deg(Z \cdot T) = \deg(W \cdot T), \tag{2.5}$$

for any algebraic cycle T on X. Here, by $\deg(V)$ we mean the degree of the algebraic cycle $V = \sum_{\text{finite}} m_\alpha V_\alpha \in C^*(X)$.

The degree defines a homomorphism from the free abelian group of algebraic cycles $C^*(X) = \bigoplus_i C^i(X)$ to the integers. On the components $C^i(X)$, the map is defined as follows

$$\deg\colon C^i(X) \to \mathbb{Z}, \quad \deg(V) = \begin{cases} \sum_\alpha m_\alpha & \text{if } i = \dim X, \\ 0 & \text{if } i < \dim X. \end{cases}$$

The symbol '$\cdot$' in (2.5) refers to the intersection product structure on $C^*(X)$, which is well-defined under the assumption of proper intersection. If $Z \cap T$ is proper (*i.e.* $\mathrm{codim}(Z \cap T) = \mathrm{codim}(Z) + \mathrm{codim}(T)$), then intersection theory supplies the definition of an intersection cycle $Z \cdot T \in C^*(X)$. Moreover, the intersection product is commutative and associative whenever is defined.

Passing from the free abelian group $C^*(X)$ to the quotient $C^*(X)/ \sim$, modulo a suitable equivalence relation on cycles, allows one to use classical results of algebraic geometry (so called Moving Lemmas) which lead to the definition of a ring structure. One then defines intersection cycle classes in general, even when cycles do not intersect properly, by intersecting equivalent cycles which fulfill the required geometric property of proper intersection.

It is natural to guess that the use of the numerical equivalence in the original definition of the category of motives was motivated by the study of classical constructions in enumerative geometry, such as for example the computation of the number of the rational points of an algebraic variety defined over a finite field. One of the main original goals was to show that for a suitable definition of an equivalence relation on algebraic cycles, the corresponding category of motives is *semi-simple*. This means that the objects M in the category decompose, following the rules of a *theory of weights* (*cf.* Section 2.3), into direct *factors* $M = \bigoplus_i M_i(X)$, with $M_i(X)$ simple (*i.e.* indecomposable) motives associated to smooth, projective algebraic varieties. The importance of achieving such a result is quite evident if one seeks, for example, to understand categorically the decomposition $H^*(X) = \bigoplus_i H^i(X)$ in cohomology, or if one wants to recognize the role of motives in the factorization of zeta-functions of algebraic varieties.

Grothendieck concentrated his efforts on the numerical equivalence relation which is the coarsest among the equivalence relations on algebraic cycles. So doing, he attacked the problem of the semi-simplicity of the category of motives from the easiest side. However, despite a promising departing point, the statement on the semi-simplicity escaped all his efforts. In fact, the result he was able to reach at that time was dependent on the assumption of the *Standard Conjectures*, two strong topological statements on algebraic cycles. The proof of the semi-simplicity of the category of motives for numerical equivalence (the only equivalence relation producing this result) was achieved only much later on in the development of the theory (*cf.* [25]). The proof found by U. Jannsen uses a fairly elementary but ingenious idea which mysteriously eluded Grothendieck's intuition as well as all the mental grasps of several mathematicians after him.

By looking at the construction of (pure) motives in perspective, one immediately recognizes the predominant role played by the morphisms over the objects, in the category $\mathbf{M}(k)_{\mathbb{Q}}$. This was certainly a great intuition of Grothendieck. This idea led to a systematic study of the properties of algebraic cycles and their decomposition by means of *algebraic projectors*, that is algebraic cycles classes $p \in A^{\dim X}(X \times X)$ satisfying the property

$$p^2 = p \circ p = p.$$

Notice that in order to make sense of the notion of a projector and more in general, in order to define a law of composition '$\circ$' on algebraic correspondences, one needs to use the ring structure on the graded algebra A^*. The operation '$\circ$' is defined as follows. Let us assume for simplicity, that the algebraic varieties are connected (the general case can be easily deduced from this). Then, two algebraic correspondences $f_1 \in A^{\dim X_1 + i}(X_1 \times X_2)$ (of degree i) and $f_2 \in A^{\dim X_2 + j}(X_2 \times X_3)$ (of degree j) compose accordingly to the following rule (bi-linear, associative)

$$A^{\dim X_1 + i}(X_1 \times X_2) \times A^{\dim X_2 + j}(X_2 \times X_3) \to A^{\dim X_1 + i + j}(X_1 \times X_3)$$

$$(f_1, f_2) \mapsto f_2 \circ f_1 = (p_{13})_*((p_{12}^*(f_1) \cdot (p_{23})^*(f_2))).$$

In the particular case of projectors $p\colon X \dashrightarrow X$, one is restricted, in order to make a sense of the condition $p \circ p = p$, to use only particular types of algebraic correspondences: namely those of degree zero. These are the elements of the abelian group $A^{\dim X}(X \times X)$.

The objects of the category $\mathbf{M}(k)_{\mathbb{Q}}$ are then pairs (X, p), with $X \in \mathrm{Obj}(\mathcal{V}_k)$ and p a projector. This way, one attains the notion of a $\mathbb{Q}$-linear, pseudo-abelian, monoidal category (the $\otimes$-monoidal structure is deduced from the cartesian product of algebraic varieties), together with the definition of a contravariant functor

$$h\colon \mathcal{V}_k \to \mathbf{M}(k)_{\mathbb{Q}}, \quad X \mapsto h(X) = (X, \mathrm{id}).$$

Here (X, id) denotes the motive associated to X and id means the trivial (*i.e.* identity) projector associated to the diagonal Δ_X. More in general, (X, p) refers to the motive $ph(X)$ that is cut-off on $h(X)$ by the (range of the) projector $p\colon X \dashrightarrow X$. Notice that images of projectors are formally included among the objects of $\mathbf{M}(k)_{\mathbb{Q}}$, by the procedure of the pseudo-abelian envelope. The cut-off performed by a projector p on the space determines a corresponding operation in cohomology (for any classical Weil theory), by singling out the sub-vector space $pH^*(X) \subset H^*(X)$.

The category $\mathbf{M}(k)_{\mathbb{Q}}$ has two important basic objects: $\mathbf{1}$ and $\mathbf{L}$. $\mathbf{1}$ is the *unit motive*

$$\mathbf{1} = (\mathrm{Spec}(k), \mathrm{id}) = h(\mathrm{Spec}(k)).$$

This is defined by the zero-dimensional algebraic variety associated to a point, whereas

$$\mathbf{L} = (\boldsymbol{P}^1, \pi_2), \quad \pi_2 = \boldsymbol{P}^1 \times \{P\}, \quad P \in \boldsymbol{P}^1(k)$$

is the so-called *Lefschetz motive*. This motive determines, jointly with $\mathbf{1}$, a decomposition of the motive associated to the projective line $\boldsymbol{P}^1$

$$h(\boldsymbol{P}^1) = \mathbf{1} \oplus \mathbf{L}. \tag{2.6}$$

One can show that the algebraic cycles $\boldsymbol{P}^1 \times \{P\}$ and $\{P\} \times \boldsymbol{P}^1$ on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ do not depend on the choice of the rational point $P \in \boldsymbol{P}^1(k)$ and that their sum is a cycle equivalent to the diagonal. This fact implies that the decomposition (2.6) is canonical. More in general, it follows from the Künneth decomposition of the diagonal Δ in $\boldsymbol{P}^n \times \boldsymbol{P}^n$ by algebraic cycles $\Delta = \pi_0 + \cdots + \pi_n$, (*cf.* [32] and [19] for the details) that the motive of a projective space $\boldsymbol{P}^n$ decomposes into pieces (simple motives)

$$h(\boldsymbol{P}^n) = h^0(\boldsymbol{P}^n) \oplus h^2(\boldsymbol{P}^n) \oplus \cdots \oplus h^{2n}(\boldsymbol{P}^n) \tag{2.7}$$

where $h^{2i}(\boldsymbol{P}^n) = (\boldsymbol{P}^n, \pi_{2i}) = (h^2(\boldsymbol{P}^n))^{\otimes i}$ for all $i > 0$. It is precisely this decomposition which implies the decomposition (2.3) on the rational points, when $k = \mathbb{F}_q$!

For (irreducible) curves, and in the presence of a rational point $x \in C(k)$, one obtains a similar decomposition (non canonical)

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C)$$

with $h^0(C) = (C, \pi_0 = \{x\} \times C)$, $h^2(C) = (C, \pi_2 = C \times \{x\})$ and $h^1(C) = (C, 1 - \pi_0 - \pi_2)$. This decomposition is responsible for the formula (2.4).

In fact, one can prove that these decompositions partially generalize to any object $X \in \mathrm{Obj}(\mathcal{V}_k)$. In the presence of a rational point, or more in general by choosing a positive zero-cycle $Z = \sum_\alpha m_\alpha Z_\alpha \in C^{\dim X}(X \times X)$ (here X is assumed irreducible for simplicity and $\dim X = d$), one constructs two rational algebraic cycles

$$\pi_0 = \frac{1}{m}(Z \times X), \quad \pi_{2d} = \frac{1}{m}(X \times Z); \quad m = \deg(Z) = \sum_\alpha m_\alpha > 0$$

which determine two projectors π_0, π_{2d} in the Chow group $\mathrm{CH}^d(X \times X) \otimes \mathbb{Q}$ of rational algebraic cycles modulo rational equivalence. The corresponding classes in

$A^d(X \times X)$ (if not zero) determine two motives $(X, \pi_0) \simeq h^0(X)$ and $(X, \pi_{2d}) \simeq h^{2d}(X)$ (*cf. e.g.* [35]).

For the applications, it is convenient to enlarge the category of effective motives by formally adding the tensor product inverse $\mathbf{L}^{-1}$ of the Lefschetz motive: one usually refers to it as to the *Tate motive*. It corresponds, from the more refined point of view of Galois theory, to the cyclotomic characters. This enlargement of $\mathbf{M}(k)_{\mathbb{Q}}$ by the so-called "virtual motives" produces an abelian, semi-simple category $\mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}}$ of pure motives for numerical equivalence. The objects of this category are now triples (X, p, m), with $m \in \mathbb{Z}$. Effective motives are of course objects of this category and they are described by triples $(X, p, 0)$. The Lefschetz motive gains a new interpretation in this category as $\mathbf{L} = (\mathrm{Spec}(k), \mathrm{id}, -1)$. The Tate motive is defined by $\mathbf{L}^{-1} = (\mathrm{Spec}(k), \mathrm{id}, 1)$ and is therefore reminiscent of (in fact induces) the notion of Tate structure $\mathbb{Q}(1)$ in Hodge theory.

In the category $\mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}}$, the set of morphisms connecting two motives (X, p, m) and (Y, q, n) is defined by

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = q \circ A^{\dim X - m + n}(X \times Y) \circ p.$$

In particular, for all $f = f^2 \in \mathrm{End}((X, p, m))$, one defines the two motives

$$\mathrm{Im}(f) = (X, p \circ f \circ p, m), \quad \mathrm{Ker}(f) = (X, p - f, m).$$

These determine a canonical decomposition of any virtual motive as $\mathrm{Im}(f) \oplus \mathrm{Im}(1 - f) \xrightarrow{\sim} (X, p, m)$, where the direct sum of two motives as the above ones is defined by the formula

$$(X, p, m) \oplus (Y, q, m) = \big(X \coprod Y, p + q, m\big).$$

The general definition of the direct sum of two motives requires a bit more of formalism which escapes this short overview: we refer to *op.cit.* for the details.

The tensor structure

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n)$$

and the involution (*i.e.* auto-duality) which is defined, for X irreducible and $\dim X = d$ by the functor

$$\vee \colon (\mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}})^{\mathrm{op}} \to \mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}}, \quad (X, p, m)^{\vee} = (X, p^t, d - m)$$

(the general case follows from this by applying additivity), determine the structure of a *rigid* $\otimes$*-category* on $\mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}}$. Here, p^t denotes the transpose correspondence associated to p (*i.e.* the transpose of the graph). One finds, for example, that $\mathbf{L}^{\vee} = \mathbf{L}^{-1}$. In the particular case of the effective motive $h(X)$, with X irreducible and $\dim X = d$, this involution determines the notion of Poincaré duality

$$h(X)^{\vee} = h(X) \otimes \mathbf{L}^{\otimes(-d)}$$

which is an auto-duality that induces the Poincaré duality isomorphism in any classical cohomological theory.

2.3 Fundamental structures

A category of pure motives over a field k and with coefficients in a field K (of characteristic zero) is supposed to satisfy, to be satisfactory, several basic properties and to be endowed with a few fundamental structures. In the previous section we have described the historically first example of a category of pure motives and we have reviewed some of its basic properties (in that case $K = \mathbb{Q}$). One naturally wonders about the description of other categories of motives associated to finer (than the numerical) equivalence relations on algebraic cycles: namely the categories of motives for homological or rational or algebraic equivalence relations. However, if one seeks to work with a semi-simple category, the afore mentioned result of Jannsen tells us that the numerical equivalence is the only adequate relation. The semi-simplicity property is also attained if one assumes Grothendieck's Standard Conjectures. Following the report of Grothendieck in [21], these conjectures arose from the hope to prove the conjectures of Weil on the zeta-function of an algebraic variety defined over a finite field. It was well known to Grothendieck that the Standard Conjectures imply the Weil's Conjectures. These latter statements became a theorem in the early seventies (1974), only a few years later the time when Grothendieck stated the Standard Conjectures (1968–69). The proof by Deligne of the Weil's conjectures, however, does not make any use of the Standard Conjectures and these latter questions remain still unanswered at the present time. The moral lesson seems to be that geometric topology and the theory of algebraic cycles govern in many central aspects the foundations of algebraic geometry.

The Standard Conjectures of "Lefschetz type" and of "Hodge type" are stated in terms of algebraic cycles modulo homological equivalence (*cf.* [27]). They imply two further important conjectures. One of these states the equality of the homological and the numerical equivalence relations, the other one, of "Künneth type", claims that the Künneth decomposition of the diagonal in cohomology can be described by means of (rational) algebraic cycles. Nowadays it is generally accepted to refer to the full set of the four conjectures, when one quotes the Standard Conjectures. In view of their expected consequences, one is naturally led to study a category of pure *motives for homological equivalence*. In fact, there are several candidates for this category since the definition depends upon the choice of a Weil cohomological theory (*i.e.* Betti, étale, de Rham, crystalline, etc.) with coefficients in a field K of characteristic zero.

Let us fix a cohomological theory $X \mapsto H^*(X) = H^*(X, K)$ for algebraic varieties in the category $\mathcal{V}_k$. Then, the construction of the corresponding category of motives $\mathcal{M}_k^{\mathrm{hom}}(\mathcal{V}_k)_K$ for homological equivalence is given by following a procedure similar to the one we have explained earlier on in this paper for the category of motives for numerical equivalence and with rational coefficients. The only difference is that now morphisms in the category $\mathcal{M}_k^{\mathrm{hom}}(\mathcal{V}_k)_K$ are defined by means of algebraic correspondences modulo homological equivalence. At this point, one makes explicit use of the axiom "cycle map" which characterizes (together with finiteness, Poincaré duality, Künneth formula and weak and strong Lefschetz theorems) any Weil cohomological theory (*cf.* [27]).

The set of algebraic morphisms connecting objects in the category $\mathcal{V}_k$ is enlarged by including multi-valued maps $X \dashrightarrow Y$ that are defined as a K-linear combination of elements of the vector spaces

$$C^*(X \times Y) \otimes_{\mathbb{Z}} F/ \sim_{\text{hom}},$$

where $F \subset K$ is a subfield. Two cycles $Z, W \in C^*(X \times Y) \otimes F$ are *homologically equivalent* $Z \sim_{\text{hom}} W$ if their image, by means of the cycle class map

$$\gamma\colon C^*(X \times Y) \otimes F \to H^*(X \times Y)$$

is the same. This leads naturally to the definition of a subvector-space $A^*_{\text{hom}}(X \times Y) \subset H^*(X \times Y)$ generated by the image of the cycle class map γ. These spaces define the correspondences in the category $\mathcal{M}_k^{\text{hom}}(\mathcal{V}_k)_K$. If X is purely d-dimensional, then

$$\operatorname{Corr}^r(X, Y) := A^{d+r}_{\text{hom}}(X \times Y).$$

In general, if X decomposes into several connected components $X = \coprod_i X_i$, one sets $\operatorname{Corr}^r(X, Y) = \bigoplus_i \operatorname{Corr}^r(X_i, Y)$. In direct analogy to the construction of correspondences for numerical equivalence, the ring structure (“cap-product”) in cohomology determines a composition law ‘$\circ$’ among correspondences.

The category $\mathcal{M}_k^{\text{hom}}(\mathcal{V}_k)_K$ is then defined as follows: the objects are triples $M = (X, p, m)$, where $X \in \operatorname{Obj}(\mathcal{V}_k)$, $m \in \mathbb{Z}$ and $p = p^2 \in \operatorname{Corr}^0(X, X)$ is an idempotent. The collection of morphisms between two motives $M = (X, p, m)$, $N = (Y, q, n)$ is given by the set

$$\operatorname{Hom}(M, N) = q \circ \operatorname{Corr}^{n-m}(X, Y) \circ p.$$

This procedure determines a pseudo-abelian, K-linear tensor category. The tensor law is given by the formula

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \times q, m + n).$$

The commutativity and associativity constraints are induced by the obvious isomorphisms $X \times Y \xrightarrow{\sim} Y \times X$, $X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$. The unit object in the category is given by $\mathbf{1} = (\operatorname{Spec}(k), \operatorname{id}, 0)$. One shows that $\mathcal{M}_k^{\text{hom}}(\mathcal{V}_k)_K$ is a *rigid* category, as it is endowed with an auto-duality functor

$$\vee\colon \mathcal{M}_k^{\text{hom}}(\mathcal{V}_k)_K \to (\mathcal{M}_k^{\text{hom}}(\mathcal{V}_k)_K)^{\text{op}}.$$

For any object M, the functor $- \otimes M^\vee$ is left-adjoint to $- \otimes M$ and $M^\vee \otimes -$ is right-adjoint to $M \otimes -$. In the case of an irreducible variety X, the internal Hom is defined by the motive

$$\underline{\operatorname{Hom}}((X, p, m), (Y, q, n)) = (X \times Y, p^t \times q, \dim(X) - m + n).$$

The Standard Conjecture of Künneth type (which is assumed from now on in this section) implies that the Künneth components of the diagonal $\pi_X^i \in \operatorname{Corr}^0(X, X)$ determine a complete system of orthogonal, central idempotents. This important statement implies that the motive $h(X) \in \mathcal{M}_k^{\text{hom}}(\mathcal{V}_k)_K$ has the expected direct sum decomposition (unique)

$$h(X) = \bigoplus_{i=0}^{2\dim X} h^i(X), \quad h^i(X) = \pi_X^i h(X).$$

The cohomology functor $X \mapsto H^*(X)$ factors through the projection $h(X) \to h^i(X)$. More in general, one shows that every motive $M = (X, p, m)$ gets this way a $\mathbb{Z}$-*grading* structure by setting

$$(X, p, m)^r = (X, p \circ \pi^{r+2m}, r). \tag{2.8}$$

This grading is respected by all morphisms in the category and defines the structure of a *graduation by weights* on the objects. On a motive $M = (X, p, m) = ph(X) \otimes \mathbf{L}^{\otimes(-m)} = ph(X)(m)$ in the category, one sets

$$M = \bigoplus_i \mathrm{Gr}_i^w(M), \quad \mathrm{Gr}_i^w(M) = ph^{2m+i}(X)(m)$$

where $\mathrm{Gr}_i^w(M)$ is a pure motive of weight i. One finds, for example, that $\mathbf{1}$ has weight zero, $\mathbf{L} = (\mathrm{Spec}(k), 1, -1)$ has weight 2 and that $\mathbf{L}^{-1} = (\mathrm{Spec}(k), 1, 1)$ has weight -2. More in general, the motive $M = (X, p, m) = ph(X)(m)$ has weight $-2m$.

In order to achieve further important properties, one needs to modify the natural commutativity constraint $\psi = \bigoplus_{r,s} \psi^{r,s}$, $\psi^{r,s} : M^r \otimes N^s \xrightarrow{\sim} N^s \otimes M^r$, by defining

$$\psi_{\mathrm{new}} = \bigoplus_{r,s} (-1)^{rs} \psi^{r,s}. \tag{2.9}$$

We shall denote by $\widetilde{\mathcal{M}}_k^{\mathrm{hom}}(\mathcal{V}_k)_K$ the category of motives for homological equivalence in which one has implemented the modification (2.9) on the tensor product structure.

An important structure on a category of pure motives (for homological equivalence) is given by assigning to an object $X \in \mathrm{Obj}(\mathcal{M}_k^{\mathrm{hom}}(\mathcal{V}_k)_K)$ a *motivic cohomology* $H^i_{\mathrm{mot}}(X)$. $H^i_{\mathrm{mot}}(X)$ is a pure motive of weight i. This way, one views pure motives as a universal cohomological theory for algebraic varieties. The main property of the motivic cohomology is that it defines a universal realization of any given Weil cohomology theory H^*. Candidates for these motivic cohomology theories have been proposed by A. Beilinson [3] in terms of eigenspaces of Adams operations in algebraic K-theory, *i.e.*, $H^{2j-n}_{\mathrm{mot}}(X, \mathbb{Q}(j)) = K_n(X)^{(j)}$, and by S. Bloch [5] in terms of higher Chow groups, *i.e.*, $H^{2j-n}_{\mathrm{mot}}(X, \mathbb{Q}(j)) = \mathrm{CH}^j(X, n) \otimes \mathbb{Q}$.

The assignment of a Weil cohomological theory with coefficients in a field K which contains an assigned field F is equivalent to the definition of an exact *realization $\otimes$-functor* of $\widetilde{\mathcal{M}}_k^{\mathrm{hom}}(\mathcal{V}_k)_K$ in the category of K-vector spaces

$$r_{H^*} : \widetilde{\mathcal{M}}_k^{\mathrm{hom}}(\mathcal{V}_k)_K \to \mathrm{Vect}_K, \quad r_{H^*}(H^i_{\mathrm{mot}}(X)) \simeq H^i(X). \tag{2.10}$$

In particular, one obtains the realization $r_{H^*}(\mathbf{L}^{-1}) = H^2(\boldsymbol{P}^1)$ which defines the notion of the *Tate twist* in cohomology. More precisely we have

– in étale cohomology: $H^2(\boldsymbol{P}^1) = \mathbb{Q}_\ell(-1)$, where $\mathbb{Q}_\ell(1) := \varprojlim_m \mu_{\ell^m}$ is a $\mathbb{Q}_\ell$-vector space of dimension one endowed with the cyclotomic action of the absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$. The "twist" (or torsion) (r) in étale cohomology corresponds to the torsion in Galois theory defined by the r-th power of the cyclotomic character (Tate twist);

– in de Rham theory: $H^2_{\mathrm{DR}}(\boldsymbol{P}^1) = k$, with the Hodge filtration defined by $F^{\leq 0} = 0$, $F^{>0} = k$. Here, the effect of the torsion (r) is that of shifting the Hodge filtration of $-r$-steps (to the right);

– in Betti theory: $H^2(\boldsymbol{P}^1) = \mathbb{Q}(-1) := (2\pi i)^{-1}\mathbb{Q}$. The bi-graduation on $H^2(\boldsymbol{P}^1) \otimes \mathbb{C} \simeq \mathbb{C}$ is purely of type $(1,1)$. The torsion (r) is here identified with the composite of a homothety given by a multiplication by $(2\pi i)^{-r}$ followed by a shifting by $(-r,-r)$ of the Hodge bi-graduation.

Using the structure of rigid tensor-category one introduces the notion of *rank* associated to a motive $M = (X, p, m)$ in $\widetilde{\mathcal{M}}_k^{\mathrm{hom}}(\mathcal{V}_k)_K$. The rank of M is defined as the trace of id_M *i.e.* the trace of the morphism $\varepsilon \circ \psi_{\mathrm{new}} \circ \eta \in \mathrm{End}(\mathbf{1})$, where

$$\varepsilon\colon M \otimes M^\vee \to \mathbf{1}, \quad \eta\colon \mathbf{1} \to M^\vee \otimes M$$

are *resp.* the evaluation and co-evaluation morphisms satisfying $\varepsilon \otimes \mathrm{id}_M \circ \mathrm{id}_M \otimes \eta = \mathrm{id}_M$, $\mathrm{id}_{M^\vee} \otimes \varepsilon \circ \eta \otimes \mathrm{id}_{M^\vee} = \mathrm{id}_{M^\vee}$. In general, one sets

$$\mathrm{rk}(X, p, m) = \sum_{i \geq 0} \dim pH^i(X) \geq 0.$$

Under the assumption of the Standard Conjectures (more precisely under the assumption that homological and numerical equivalence relations coincide) and that $\mathrm{End}(\mathbf{1}) = F$ ($\mathrm{char}(F) = 0$), the tannakian formalism invented by Grothendieck and developed by Saavedra [34], and Deligne [17] implies that the abelian, rigid, semi-simple tensor category $\widetilde{\mathcal{M}}_k^{\mathrm{hom}}(\mathcal{V}_k)_K$ is endowed with an exact, faithful $\otimes$-fibre functor to the category of graded K-vector spaces

$$\omega\colon \widetilde{\mathcal{M}}_k^{\mathrm{hom}}(\mathcal{V}_k)_K \to \mathrm{VectGr}_K, \quad \omega(H^*_{\mathrm{mot}}(X)) = H^*(X) \tag{2.11}$$

which is compatible with the realization functor. This formalism defines a *tannakian* (*neutral* if $K = F$) structure on the category of motives. One then introduces the *tannakian group*

$$G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$$

as a K-scheme in affine groups. Through the tannakian formalism one shows that the fibre functor ω realizes an equivalence of rigid tensor categories

$$\omega\colon \widetilde{\mathcal{M}}_k^{\mathrm{hom}}(\mathcal{V}_k)_K \xrightarrow{\sim} \mathrm{Rep}_F(G),$$

where $\mathrm{Rep}_F(G)$ denotes the rigid tensor category of finite dimensional, F representations of the tannakian group G. This way, one establishes a quite useful dictionary between categorical $\otimes$-properties and properties of the associated groups. Because we have assumed all along the Standard Conjectures, the semi-simplicity of the category $\widetilde{\mathcal{M}}_k^{\mathrm{hom}}(\mathcal{V}_k)_K$ implies that G is an algebraic, pro-reductive group, that is G is the projective limit of reductive F-algebraic groups.

The tannakian theory is a linear analog of the theory of finite, étale coverings of a given connected scheme. This theory was developed by Grothendieck in SGA1 (theory of the pro-finite π_1). For this reason the group G is usually referred to as the *motivic Galois group* associated to $\mathcal{V}_k$ and H^*. In the case of algebraic varieties of dimension zero (*i.e.* for Artin motives) the tannakian group G is nothing but the (absolute) Galois group $\mathrm{Gal}(\bar{k}/k)$.

In any reasonable cohomological theory the functors $X \mapsto H^*(X)$ are deduced by applying standard methods of homological algebra to the related derived functors $X \mapsto R\Gamma(X)$ which associate to an object in $\mathcal{V}_k$ a bounded complex of k-vector spaces, in a suitable triangulated category $\mathcal{D}(k)$ of complexes of modules over k, whose heart is the category of motives. This is the definition of cohomology as

$$H^i(X) = H^i R\Gamma(X).$$

Under the assumption that the functors $R\Gamma$ are realizations of corresponding motivic functors *i.e.* $R\Gamma = r_{H^*} R\Gamma_{\mathrm{mot}}$, one expects the existence of a (non-canonical) isomorphism in $\mathcal{D}(k)$

$$R\Gamma_{\mathrm{mot}}(X) \simeq \bigoplus_i H^i_{\mathrm{mot}}(X)[-i]. \tag{2.12}$$

Moreover, the introduction of the motivic derived functors $R\Gamma_{\mathrm{mot}}$ suggests the definition of the following groups of *absolute cohomology*

$$H^i_{\mathrm{abs}}(X) = \mathrm{Hom}_{\mathcal{D}(k)}(\mathbf{1}, R\Gamma_{\mathrm{mot}}(X)[i]).$$

For a general motive $M = (X, p, m)$, one defines

$$H^i_{\mathrm{abs}}(M) = \mathrm{Hom}_{\mathcal{D}(k)}(\mathbf{1}, M[i]) = \mathrm{Ext}^i(\mathbf{1}, M). \tag{2.13}$$

The motives $H^i_{\mathrm{mot}}(X)$ and the groups of absolute motivic cohomology are related by a spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{abs}}(H^q_{\mathrm{mot}}(X)) \Rightarrow H^{p+q}_{\mathrm{abs}}(X).$$

2.4 Examples of pure motives

The first interesting examples of pure motives arise by considering the category $\mathcal{V}_k^o$ of étale, finite k-schemes. An object in this category is a scheme $X = \mathrm{Spec}(k')$, where k' is a commutative k-algebra of finite dimension which satisfies the following properties. Let $\bar{k}$ denote a fixed separable closure of k.

1. $k' \otimes \bar{k} \simeq \bar{k}^{[k':k]}$;
2. $k' \simeq \prod k_\alpha$, for k_α/k finite, separable field extensions;
3. $|X(\bar{k})| = [k' : k]$.

The corresponding rigid, tensor-category of motives with coefficients in a field K is usually referred to as the category of *Artin motives*: $\mathcal{C}\mathcal{V}^o(k)_K$.

The definition of this category is independent of the choice of the equivalence relation on cycles as the objects of $\mathcal{V}_k^o$ are smooth, projective k-varieties of dimension zero. One also sees that passing from $\mathcal{V}_k^o$ to $\mathcal{CV}^o(k)_K$ requires adding new objects in order to attain the property that the category of motives is abelian. One can verify this already for $k = K = \mathbb{Q}$, by considering the real quadratic extension $k' = \mathbb{Q}(\sqrt{2})$ and the one-dimensional non-trivial representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ that factors through the character of order two of $\mathrm{Gal}(k'/\mathbb{Q})$. This representation does not correspond to any object in $\mathcal{V}_{\mathbb{Q}}^o$, but can be obtained as the image of the projector $p = \frac{1}{2}(1-\sigma)$, where σ is the generator of $\mathrm{Gal}(k'/\mathbb{Q})$. Therefore, $\mathrm{image}(p) \in \mathrm{Obj}(\mathcal{CV}^o(\mathbb{Q})_{\mathbb{Q}})$ is a new object.

The category of Artin motives is a semi-simple, K-linear, monoidal $\otimes$-category. When $\mathrm{char}(K) = 0$, the commutative diagram of functors

$$\begin{tikzcd}
\mathcal{V}_k^o \arrow[r, "\mathrm{GG}", "\sim"'] \arrow[d, "h"'] & \{\text{sets with } \mathrm{Gal}(\bar{k}/k)\text{-continuous action}\} \arrow[d, "l"] \\
\mathcal{CV}^o(k)_K \arrow[r, "(*)", "\sim"'] & \{\text{finite dim. } K\text{-v.spaces with linear } \mathrm{Gal}(\bar{k}/k)\text{-continuous action}\}
\end{tikzcd}$$

where l is the contravariant functor of linearization

$$S \mapsto K^S, \quad (g(f))(s) = f(g^{-1}(s)) \quad \text{for all } g \in \mathrm{Gal}(\bar{k}/k),$$

determines a linearization of the Galois–Grothendieck correspondence (GG) by means of the equivalence of categories $(*)$. This is provided by the fiber functor

$$\omega \colon X \to H^0(X_{\bar{k}}, K) = K^{X(\bar{k})}$$

and by applying the tannakian formalism. It follows that $\mathcal{CV}^o(k)_K$ is $\otimes$-equivalent to the category $\mathrm{Rep}_K \mathrm{Gal}(\bar{k}/k)$ of representations of the absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$.

These results were the departing point for Grothendieck's speculations on the definition of higher dimensional Galois theories (*i.e.* Galois theories associated to system of polynomials in several variables) and for the definition of the corresponding motivic Galois groups.

3 Endomotives: an overview

The notion of an *endomotive* in noncommutative geometry (*cf.* [10]) is the natural generalization of the classical concept of an Artin motive for the noncommutative spaces which are defined by semigroup actions on projective limits of zero-dimensional algebraic varieties, endowed with an action of the absolute Galois group. This notion applies quite naturally for instance, to the study of several examples of quantum statistical dynamical systems whose time evolution describes important number-theoretic properties of a given field k (*cf. op.cit*, [15]).

There are two distinct definitions of an endomotive: one speaks of algebraic or analytic endomotives depending upon the context and the applications.

When k is a number field, there is a functor connecting the two related categories. Moreover, the abelian category of Artin motives embeds naturally as a full subcategory in the category of algebraic endomotives (*cf.* Theorem 5.3) and this result motivates the statement that the theory of endomotives defines a natural generalization of the classical theory of (zero-dimensional) Artin motives.

In noncommutative geometry, where the properties of a space (frequently highly singular from a classical viewpoint) are analyzed in terms of the properties of the associated noncommutative algebra and its (space of) irreducible representations, it is quite natural to look for a suitable abelian category which enlarges the original, non-additive category of algebras and in which one may also apply the standard techniques of homological algebra. Likewise in the construction of a theory of motives, one seeks to work within a triangulated category endowed with several structures. These include for instance, the definition of (noncommutative) motivic objects playing the role of motivic cohomology (*cf.* (2.12)), the construction of a universal (co)homological theory representing in this context the absolute motivic cohomology (*cf.* (2.13)) and possibly also the set-up of a noncommutative tannakian formalism to motivate in rigorous mathematical terms the presence of certain universal groups of symmetries associated to renormalizable quantum field theories (*cf. e.g.* [13]).

A way to attack these problems is that of enlarging the original category of algebras and morphisms by introducing a "derived" category of modules enriched with a suitable notion of correspondences connecting the objects that should also account for the structure of Morita equivalence which represents the noncommutative generalization of the notion of isomorphism for commutative algebras.

3.1 The abelian category of cyclic modules

The sought for enlargement of the category Alg_k of (unital) k-algebras and (unital) algebra homomorphisms is defined by introducing a new category Λ_k of cyclic $k(\Lambda)$-modules. The objects of this category are modules over the *cyclic category* Λ. This latter has the same objects as the simplicial category Δ (Λ contains Δ as subcategory). We recall that an object in Δ is a totally ordered set

$$[n] = \{0 < 1 < \cdots < n\}$$

for each $n \in \mathbb{N}$, and a morphism

$$f : [n] \to [m]$$

is described by an order-preserving map of sets $f : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$. The set of morphisms in Δ is generated by faces $\delta_i : [n-1] \to [n]$ (the injection that misses i) and degeneracies $\sigma_j : [n+1] \to [n]$ (the surjection which identifies j with

$j+1$) which satisfy several standard simplicial identities (*cf. e.g.* [8]). The set of morphisms in Λ is enriched by introducing a new collection of morphisms: the *cyclic morphisms*. For each $n \in \mathbb{N}$, one sets

$$\tau_n \colon [n] \to [n]$$

fulfilling the relations

$$\begin{aligned} \tau_n \delta_i &= \delta_{i-1} \tau_{n-1} \quad \text{for } 1 \le i \le n, \quad \tau_n \delta_0 = \delta_n, \\ \tau_n \sigma_i &= \sigma_{i-1} \tau_{n+1} \quad \text{for } 1 \le i \le n, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2, \\ \tau_n^{n+1} &= 1_n. \end{aligned} \tag{3.1}$$

The objects of the category Λ_k are k-modules over Λ (*i.e.* $k(\Lambda)$-modules). In categorical language this means functors

$$\Lambda^{\text{op}} \to \text{Mod}_k$$

($\text{Mod}_k =$ category of k-modules). Morphisms of $k(\Lambda)$-modules are therefore natural transformations between the corresponding functors.

It is evident that Λ_k is an abelian category, because of the interpretation of a morphism in Λ_k as a collection of k-linear maps of k-modules $A_n \to B_n$ ($A_n, B_n \in \text{Obj}(\text{Mod}_k)$) compatible with faces, degeneracies and cyclic operators. Kernels and cokernels of these morphisms define objects of the category Λ_k, since their definition is given point-wise.

To an algebra $\mathcal{A}$ over a field k, one associates the $k(\Lambda)$-module $\mathcal{A}^\natural$. For each $n \ge 0$ one sets

$$\mathcal{A}_n^\natural = \underbrace{\mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{(n+1)\text{-times}}.$$

The *cyclic morphisms* on $\mathcal{A}^\natural$ correspond to the correspond to the cyclic permutations of the tensors, while the face and the degeneracy maps correspond respectively to the algebra product of consecutive tensors and to the insertion of the unit. This construction determines a functor

$$\natural \colon \text{Alg}_k \to \Lambda_k.$$

The mapping trace $\varphi \colon \mathcal{A} \to k$ gives rise to

$$\varphi^\natural \colon \mathcal{A}^\natural \to k^\natural, \quad \varphi^\natural(a_0 \otimes \cdots \otimes a_n) = \varphi(a_0 \cdots a_n)$$

in Λ_k. The main result of this construction is the following canonical description of the cyclic cohomology of an algebra $\mathcal{A}$ over a field k as the derived functor of the functor which assigns to a $k(\Lambda)$-module its space of traces

$$HC^n(\mathcal{A}) = \text{Ext}^n(\mathcal{A}^\natural, k^\natural) \tag{3.2}$$

(*cf.* [8], [30]). This formula is the analog of (2.13), that describes the absolute motivic cohomology group of a classical motive as an Ext-group computed in a triangulated category of motives $\mathcal{D}(k)$. In the present context, on the other hand, the derived groups Ext^n are taken in the abelian category of Λ_k-modules.

The description of the cyclic cohomology as a derived functor in the cyclic category determines a useful procedure to embed the nonadditive category of algebras and algebra homomorphisms in the "derived" abelian category of $k(\Lambda)$-modules. This construction provides a natural framework for the definition of the objects of a category of noncommutative motives.

Likewise in the construction of the category of motives, one is faced with the problem of finding the "motivated maps" connecting cyclic modules. The natural strategy is that of enlarging the collection of cyclic morphisms which are functorially induced by homomorphisms between (noncommutative) algebras, by implementing an adequate definition of (noncommutative) correspondences. The notion of an algebraic correspondence in algebraic geometry, as a multi-valued map defined by an algebraic cycle modulo a suitable equivalence relation, has here an analog with the notion of a Kasparov bimodule and the associated class in KK-theory (*cf.* [26]). Likewise in classical motive theory, one may prefer to work with (*i.e.* compare) several versions of correspondences. One may decide to retain the full information supplied by a group action on a given algebra (*i.e.* a noncommutative space) rather than partially loosing this information by moding out with the equivalence relation (homotopy in KK-theory).

3.2 Bimodules and *KK*-theory

There is a natural way to associate a cyclic morphism to a (virtual) correspondence and hence to a class in KK-theory. Starting with the category of separable C^*-algebras and $*$-homomorphisms, one enlarges the collection of morphisms connecting two unital algebras $\mathcal{A}$ and $\mathcal{B}$, by including correspondences defined by elements of Kasparov's bivariant K-theory

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = KK(\mathcal{A}, \mathcal{B})$$

([26], *cf.* also §8 and §9.22 of [4]). More precisely, correspondences are defined by Kasparov's bimodules, that means by triples

$$\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B}) = (E, \phi, F)$$

which satisfy the following conditions:

- E is a countably generated Hilbert module over $\mathcal{B}$;
- ϕ is a $*$-homomorphism of $\mathcal{A}$ to bounded linear operators on E (*i.e.* ϕ gives E the structure of an $\mathcal{A}$-$\mathcal{B}$ bimodule);
- F is a bounded linear operator on E such that the operators $[F, \phi(a)]$, $(F^2 - 1)\phi(a)$, and $(F^* - F)\phi(a)$ are compact for all $a \in \mathcal{A}$.

A Hilbert module E over $\mathcal{B}$ is a right $\mathcal{B}$-module with a *positive*, $\mathcal{B}$-valued inner product which satisfies $\langle x, yb\rangle = \langle x, y\rangle b$ for all $x, y \in E$ and all $b \in \mathcal{B}$, and with respect to which E is complete (*i.e.* complete in the norm $\| x \| = \sqrt{\| \langle x, x\rangle \|}$).

Notice that Kasparov bimodules are Morita-type of correspondences. They generalize $*$-homomorphisms of C^*-algebras since the latter ones may be re-interpreted as Kasparov bimodules of the form $(\mathcal{B}, \phi, 0)$.

Given a Kasparov bimodule $\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B})$, that is, an $\mathcal{A}$-$\mathcal{B}$ Hilbert bimodule E as defined above, one associates, under the assumption that E is a projective $\mathcal{B}$-module of finite type (*cf.* [10], Lemma 2.1), a cyclic morphism

$$\mathcal{E}^\natural \in \operatorname{Hom}(\mathcal{A}^\natural, \mathcal{B}^\natural).$$

This result allows one to define an enlargement of the collection of cyclic morphisms in the category Λ_k of $k(\Lambda)$-modules, by considering Kasparov's projective bimodules of finite type, as correspondences.

One then implements the *homotopy equivalence relation* on the collection of Kasparov bimodules. Two Kasparov modules are said to be homotopy equivalent $(E_0, \phi_0, F_0) \sim_h (E_1, \phi_1, F_1)$ if there is an element

$$(E, \phi, F) \in \mathcal{E}(\mathcal{A}, I\mathcal{B}), \quad I\mathcal{B} = \{f : [0, 1] \to \mathcal{B} \mid f \text{ continuous}\}$$

which performs a unitary homotopy deformation between the two modules. This means that $(E\hat{\otimes}_{f_i}\mathcal{B}, f_i \circ \phi, f_i(F))$ is unitarily equivalent to (E_i, ϕ_i, F_i) or equivalently re-phrased, that there is a unitary in bounded operators from $E\hat{\otimes}_{f_i}\mathcal{B}$ to E_i intertwining the morphisms $f_i \circ \phi$ and ϕ_i and the operators $f_i(F)$ and F_i. Here $f_i : I\mathcal{B} \to \mathcal{B}$ is the evaluation at the endpoints.

There is a binary operation on the set of all Kasparov $\mathcal{A}$-$\mathcal{B}$ bimodules, given by the direct sum. By definition, the group of Kasparov's bivariant K-theory is the set of homotopy equivalence classes $c(\mathcal{E}(\mathcal{A}, \mathcal{B})) \in KK(\mathcal{A}, \mathcal{B})$ of Kasparov modules $\mathcal{E}(\mathcal{A}, \mathcal{B})$. This set has a natural structure of abelian group with addition induced by direct sum.

This bivariant version of K-theory is richer than both K-theory and K-homology, as it carries an intersection product. There is a natural bi-linear, associative composition (intersection) product

$$\otimes_{\mathcal{B}} : KK(\mathcal{A}, \mathcal{B}) \times KK(\mathcal{B}, \mathcal{C}) \to KK(\mathcal{A}, \mathcal{C})$$

for all $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ separable C^*-algebras. This product is compatible with composition of morphisms of C^*-algebras.

KK-theory is also endowed with a bi-linear, associative exterior product

$$\otimes : KK(\mathcal{A}, \mathcal{B}) \otimes KK(\mathcal{C}, \mathcal{D}) \to KK(\mathcal{A} \otimes \mathcal{C}, \mathcal{B} \otimes \mathcal{D}),$$

which is defined in terms of the composition product by

$$c_1 \otimes c_2 = (c_1 \otimes 1_{\mathcal{C}}) \otimes_{\mathcal{B}\otimes\mathcal{C}} (c_2 \otimes 1_{\mathcal{B}}).$$

A slightly different formulation of KK-theory, which simplifies the definition of this external tensor product is obtained by replacing in the data (E, ϕ, F) the operator F by an *unbounded*, regular self-adjoint operator D. The corresponding F is then given by $D(1 + D^2)^{-1/2}$ (*cf.* [1]).

The above construction which produces an enlargement of the category of separable C^*-algebras by introducing correspondences as morphisms determines an additive, although non abelian category $\mathcal{KK}$ (*cf.* [4] §9.22.1). This category is also known to have a triangulated structure (*cf.* [33]) and this result is in agreement with the construction of the triangulated category $\mathcal{D}(k)$ in motives theory, whose heart is expected to be the category of (mixed) motives (*cf.* Section 2.3 and [18]). A more refined analysis based on the analogy with the construction of a category of motives suggests that one should probably perform a further enlargement by passing to the pseudo-abelian envelope of $\mathcal{KK}$, that is by formally including among the objects also ranges of idempotents in KK-theory.

In Section 5 we will review the category of analytic endomotives where maps are given in terms of étale correspondences described by spaces $\mathcal{Z}$ arising from locally compact étale groupoids $\mathcal{G} = \mathcal{G}(X_\alpha, S, \mu)$ associated to zero-dimensional, singular quotient spaces $X(\bar{k})/S$ with associated C^*-algebras $C^*(\mathcal{G})$. In view of what we have said in this section, it would be also possible to define a category where morphisms are given by classes $c(\mathcal{Z}) \in KK(C^*(\mathcal{G}), C^*(\mathcal{G}'))$ which describe sets of equivalent triples (E, ϕ, F), where (E, ϕ) is given in terms of a bimodule $\mathcal{M}_{\mathcal{Z}}$ with the trivial grading $\gamma = 1$ and the zero endomorphism $F = 0$. The definition of the category of analytic endomotives is more refined because the definition of the maps in this category does not require to divide by homotopy equivalence.

The comparison between correspondences for motives given by algebraic cycles and correspondences for noncommutative spaces given by bimodules (or elements in KK-theory) is particularly easy in the zero-dimensional case because the equivalence relations play no role. Of course, it would be quite interesting to investigate the higher dimensional cases, in view of a unified framework for motives and noncommutative spaces which is suggested, for example, by the recent results on the Lefschetz trace formula for archimedean local factors of L-functions of motives (*cf.* [10], Section 7).

A way to attack this problem is by comparing the notion of a correspondence given by an algebraic cycle with the notion of a geometric correspondence used in topology (*cf.* [2], [14]). For example, it is easy to see that the definition of an algebraic correspondence can be reformulated as a particular case of the topological (geometrical) correspondence and it is also known that one may associate to the latter a class in KK-theory. In the following two sections, we shall review and comment on these ideas.

3.3 Geometric correspondences

In geometric topology, given two smooth manifolds X and Y (it is enough to assume that X is a locally compact parameter space), a *topological* (*geometric*) *correspondence* is given by the datum

$$X \xleftarrow{f_X} (Z, E) \xrightarrow{g_Y} Y,$$

where

- Z is a smooth manifold,
- E is a complex vector bundle over Z,
- $f_X : Z \to X$ and $g_Y : Z \to Y$ are continuous maps, with f_X proper and g_Y K-oriented (orientation in K-homology).

Unlike in the definition of an algebraic correspondence (*cf.* Section 2.3) one does not require that Z is a subset of the cartesian product $X \times Y$. This flexibility is balanced by the implementation of the extra piece of datum given by the vector bundle E. To any such correspondence (Z, E, f_X, g_Y) one associates a class in Kasparov's K-theory

$$c(Z, E, f_X, g_Y) = (f_X)_*((E) \otimes_Z (g_Y)!) \in KK(X, Y). \tag{3.3}$$

(E) denotes the class of E in $KK(Z, Z)$ and $(g_Y)!$ is the element in KK-theory which fulfills the Grothendieck Riemann–Roch formula.

We recall that given two smooth manifolds X_1 and X_2 and a continuous oriented map $f : X_1 \to X_2$, the element $f! \in KK(X_1, X_2)$ determines the Grothendieck Riemann–Roch formula

$$\mathrm{ch}(F \otimes f!) = f_!(\mathrm{Td}(f) \cup \mathrm{ch}(F)), \tag{3.4}$$

for all $F \in K^*(X_1)$, with $\mathrm{Td}(f)$ the Todd genus

$$\mathrm{Td}(f) = \mathrm{Td}(TX_1)/\mathrm{Td}(f^*TX_2). \tag{3.5}$$

The composition of two correspondences (Z_1, E_1, f_X, g_Y) and (Z_2, E_2, f_Y, g_W) is given by taking the fibered product $Z = Z_1 \times_Y Z_2$ and the bundle $E = \pi_1^* E_1 \times \pi_2^* E_2$, with $\pi_i : Z \to Z_i$ the projections. This determines the composite correspondence (Z, E, f_X, g_W). In fact, one also needs to assume a *transversality* condition on the maps g_Y and f_Y in order to ensure that the fibered product Z is a smooth manifold. The homotopy invariance of both $g_Y!$ and $(f_X)_*$ show however that the assumption of transversality is 'generically' satisfied.

Theorem 3.2 of [14] shows that Kasparov product in KK-theory '$\otimes$' agrees with the composition of correspondences, namely

$$c(Z_1, E_1, f_X, g_Y) \otimes_Y c(Z_2, E_2, f_Y, g_W) = c(Z, E, f_X, g_W) \in KK(X, W). \tag{3.6}$$

3.4 Algebraic correspondences and K-theory

In algebraic geometry, the notion of correspondence that comes closest to the definition of a geometric correspondence (as an element in KK-theory) is obtained by considering classes of algebraic cycles in algebraic K-theory (*cf.* [31]).

Given two smooth and projective algebraic varieties X and Y, we denote by p_X and p_Y the projections of $X \times Y$ onto X and Y respectively and we assume that they are *proper*. Let $Z \in C^*(X \times Y)$ be an algebraic cycle. For simplicity, we shall assume that Z is irreducible (the general case follows by linearity). We denote by $f_X = p_X|_Z$ and $g_Y = p_Y|_Z$ the restrictions of p_X and p_Y to Z.

To the irreducible subvariety $T \overset{i}{\hookrightarrow} Y$ one naturally associates the coherent $\mathcal{O}_Y$-module $i_*\mathcal{O}_T$. For simplicity of notation we write it as $\mathcal{O}_T$. We use a similar notation for the coherent sheaf $\mathcal{O}_Z$, associated to the irreducible subvariety $Z \hookrightarrow X \times Y$. Then, the sheaf pullback

$$p_Y^*\mathcal{O}_T = p_Y^{-1}\mathcal{O}_T \otimes_{p_Y^{-1}\mathcal{O}_Y} \mathcal{O}_{X\times Y} \tag{3.7}$$

has a natural structure of $\mathcal{O}_{X\times Y}$-module. The map on sheaves that corresponds to the cap product by Z on cocycles is given by

$$Z : \mathcal{O}_T \mapsto (p_X)_*\big(p_Y^*\mathcal{O}_T \otimes_{\mathcal{O}_{X\times Y}} \mathcal{O}_Z\big). \tag{3.8}$$

Since p_X is proper, the resulting sheaf is coherent. Using (3.7), we can write equivalently

$$Z : \mathcal{O}_T \mapsto (p_X)_*\big(p_Y^{-1}\mathcal{O}_T \otimes_{p_Y^{-1}\mathcal{O}_Y} \mathcal{O}_Z\big). \tag{3.9}$$

We recall that the functor $f_!$ is the right adjoint to f^* (*i.e.* $f^*f_! = \mathrm{id}$) and that $f_!$ satisfies the Grothendieck Riemann–Roch formula

$$\mathrm{ch}(f_!(F)) = f_!(\mathrm{Td}(f) \cup \mathrm{ch}(F)). \tag{3.10}$$

Using this result, we can equally compute the intersection product of (3.8) by first computing

$$\mathcal{O}_T \otimes_{\mathcal{O}_Y} (p_Y)_!\mathcal{O}_Z \tag{3.11}$$

and then applying p_Y^*. Using (3.10) and (3.4) we know that we can replace (3.11) by $\mathcal{O}_T \otimes_{\mathcal{O}_Y} (\mathcal{O}_Z \otimes (p_Y)!)$ with the same effect in K-theory.

Thus, to a correspondence in the sense of (3.8) that is defined by the image in K-theory of an algebraic cycle $Z \in C^*(X \times Y)$ we associate the geometric class

$$\mathcal{F}(Z) = c(Z, E, f_X, g_Y) \in KK(X, Y),$$

with $f_X = p_X|_Z$, $g_Y = p_Y|Z$ and with the bundle $E = \mathcal{O}_Z$.

The composition of correspondences is given in terms of the intersection product of the associated cycles. Given three smooth projective varieties X, Y and W and (virtual) correspondences $U = \sum a_i Z_i \in C^*(X\times Y)$ and $V = \sum c_j Z'_j \in C^*(Y \times W)$, with $Z_i \subset X \times Y$ and $Z'_j \subset Y \times W$ closed reduced irreducible subschemes, one defines

$$U \circ V = (\pi_{13})_*((\pi_{12})^*U \cdot (\pi_{23})^*V). \tag{3.12}$$

$\pi_{12}\colon X\times Y\times W \to X\times Y$, $\pi_{23}\colon X\times Y\times W \to Y\times W$, and $\pi_{13}\colon X\times Y\times W \to X\times W$ denote, as usual, the projection maps.

Under the assumption of 'general position' which is the algebraic analog of the transversality requirement in topology, we obtain the following result

Proposition 3.1 (*cf.* [10] Proposition 6.1). *Suppose given three smooth projective varieties X, Y, and W, and algebraic correspondences U and V given by $Z_1 \subset X\times Y$*

and $Z_2 \subset Y \times W$, respectively. Assume that the cycles $(\pi_{12})^* Z_1$ and $(\pi_{23})^* Z_2$ are in general position in $X \times Y \times W$. Then assigning to a cycle Z the topological correspondence $\mathcal{F}(Z) = (Z, E, f_X, g_Y)$ satisfies

$$\mathcal{F}(Z_1 \circ Z_2) = \mathcal{F}(Z_1) \otimes_Y \mathcal{F}(Z_2), \tag{3.13}$$

where $Z_1 \circ Z_2$ is the product of algebraic cycles and $\mathcal{F}(Z_1) \otimes_Y \mathcal{F}(Z_2)$ is the Kasparov product of the topological correspondences.

Notice that, while in the topological (smooth) setting transversality can always be achieved by a small deformation (*cf.* §III, [14]), in the algebro-geometric framework one needs to modify the above construction if the cycles are not in general position. In this case the formula

$$[\mathcal{O}_{T_1}] \otimes [\mathcal{O}_{T_2}] = [\mathcal{O}_{T_1 \circ T_2}]$$

which describes the product in K-theory in terms of the intersection product of algebraic cycles must be modified by implementing Tor-classes and one works with a product defined by the formula ([31], Theorem 2.7)

$$[\mathcal{O}_{T_1}] \otimes [\mathcal{O}_{T_2}] = \sum_{i=0}^{n} (-1)^i \big[\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{T_1}, \mathcal{O}_{T_2})\big]. \tag{3.14}$$

4 Algebraic endomotives

To define the category of *algebraic endomotives* one replaces the category $\mathcal{V}_k^o$ of reduced, finite-dimensional commutative algebras (and algebra homomorphisms) over a field k by the category of noncommutative algebras (and algebra homomorphisms) of the form

$$\mathcal{A}_k = A \rtimes S.$$

A denotes a unital algebra which is an inductive limit of commutative algebras $A_\alpha \in \mathrm{Obj}(\mathcal{V}_k^o)$. S is a unital, *abelian semigroup* of algebra endomorphisms

$$\rho \colon A \to A.$$

Moreover, one imposes the condition that for $\rho \in S$, $e = \rho(1) \in A$ is an *idempotent* of the algebra and that ρ is an isomorphism of A with the compressed algebra eAe.

The crossed product algebra $\mathcal{A}_k$ is defined by formally adjoining to A new generators U_ρ and U_ρ^*, for $\rho \in S$, satisfying the algebraic rules

$$\begin{aligned} U_\rho^* U_\rho &= 1, & U_\rho U_\rho^* &= \rho(1), && \text{for all } \rho \in S,\\ U_{\rho_1\rho_2} &= U_{\rho_1} U_{\rho_2}, & U_{\rho_2\rho_1}^* &= U_{\rho_1}^* U_{\rho_2}^*, && \text{for all } \rho_j \in S,\\ U_\rho x &= \rho(x) U_\rho, & x U_\rho^* &= U_\rho^* \rho(x), && \text{for all } \rho \in S,\ x \in A. \end{aligned} \tag{4.1}$$

Since S is abelian, these rules suffice to show that $\mathcal{A}_k$ is the linear span of the monomials $U_{\rho_1}^* a U_{\rho_2}$, for $a \in A$ and $\rho_j \in S$.

Because $A = \varinjlim_\alpha A_\alpha$, with A_α reduced, finite-dimensional commutative algebras over k, the construction of $\mathcal{A}_k$ is in fact determined by assigning a *projective system* $\{X_\alpha\}_{\alpha \in I}$ of varieties in $\mathcal{V}_k^o$ (I is a countable indexing set), with $\xi_{\beta,\alpha} \colon X_\beta \to X_\alpha$ morphisms in $\mathcal{V}_k^o$ and with a suitably defined action of S. Here, we have implicitly used the equivalence between the category of finite dimensional commutative k-algebras and the category of affine algebraic varieties over k.

The graphs $\Gamma_{\xi_{\beta,\alpha}}$ of the connecting morphisms of the projective system define $G_k = \mathrm{Gal}(\bar{k}/k)$-invariant subsets of $X_\beta(\bar{k}) \times X_\alpha(\bar{k})$ which in turn describe $\xi_{\beta,\alpha}$ as algebraic correspondences. We denote by

$$X = \varprojlim_\alpha X_\alpha, \quad \xi_\alpha \colon X \to X_\alpha$$

the associated *pro-variety*. The compressed algebra eAe associated to the idempotent $e = \rho(1)$ determines a subvariety $X^e \subset X$ which is in fact isomorphic to X, via the induced morphism $\tilde{\rho} \colon X \to X^e$.

The noncommutative space defined by $\mathcal{A}_k$ is the quotient of $X(\bar{k})$ by the action of S, *i.e.* of the action of the $\tilde{\rho}$'s.

The Galois group G_k acts on $X(\bar{k})$ by composition. By identifying the elements of $X(\bar{k})$ with characters, *i.e.* with k-algebra homomorphisms $\chi \colon A \to \bar{k}$, we write the action of G_k on A as

$$\alpha(\chi) = \alpha \circ \chi \colon A \to \bar{k} \quad \text{for all } \alpha \in G_k,\ \chi \in X(\bar{k}). \tag{4.2}$$

This action commutes with the maps $\tilde{\rho}$, *i.e.* $(\alpha \circ \chi) \circ \rho = \alpha \circ (\chi \circ \rho)$. Thus the whole construction of the system (X_α, S) is G_k-equivariant. This fact does not mean, however, that G_k acts by automorphisms on $\mathcal{A}_k$!

Moreover, notice that the algebraic construction of the crossed-product algebra $\mathcal{A}_k$ endowed with the actions of G_k and S on $X(\bar{k})$ makes sense also when $\mathrm{char}(k) > 0$.

When $\mathrm{char}(k) = 0$, one defines the set of correspondences $M(\mathcal{A}_k, \mathcal{B}_k)$ by using the notion of Kasparov bimodules $\mathcal{E}(\mathcal{A}_k, \mathcal{B}_k)$ which are projective and finite as right modules. This way, one obtains a first realization of the resulting category of noncommutative zero-dimensional motives in the abelian category of $k(\Lambda)$-modules.

In general, given (X_α, S), with $\{X_\alpha\}_{\alpha \in I}$ a projective system of Artin motives and S a semigroup of endomorphisms of $X = \varprojlim_\alpha X_\alpha$ as above, the datum of the semigroup action is encoded naturally by the *algebraic groupoid*

$$\mathcal{G} = X \rtimes S.$$

This is defined in the following way. One considers the Grothendieck group $\widetilde{S}$ of the abelian semigroup S. By using the injectivity of the partial action of S, one may also assume that S embeds in $\widetilde{S}$. Then, the action of S on X extends to define a *partial* *action* of $\widetilde{S}$. More precisely, for $s = \rho_2^{-1}\rho_1 \in \widetilde{S}$ the two projections

$$E(s) = \rho_1^{-1}(\rho_2(1)\rho_1(1)), \quad F(s) = \rho_2^{-1}(\rho_2(1)\rho_1(1))$$

only depend on s and the map $s\colon A_{E(s)} \to A_{F(s)}$ defines an isomorphism of reduced algebras. It is immediate to verify that $E(s^{-1}) = F(s) = s(E(s))$ and that $F(ss') \geq F(s)s(F(s'))$. The algebraic groupoid $\mathcal{G}$ is defined as the disjoint union

$$\mathcal{G} = \coprod_{s \in \widetilde{S}} X^{F(s)}$$

which corresponds to the commutative direct-sum of reduced algebras

$$\bigoplus_{s \in \widetilde{S}} A_{F(s)}.$$

The range and the source maps in $\mathcal{G}$ are given *resp.* by the natural projection from $\mathcal{G}$ to X and by its composition with the antipode $\boldsymbol{S}$ which is defined, at the algebra level, by $\boldsymbol{S}(a)_s = s(a_{s^{-1}})$ for all $s \in \widetilde{S}$. The composition in the groupoid corresponds to the product of monomials $aU_s bU_t = as(b)U_{st}$.

Given two systems (X_α, S) and $(X'_{\alpha'}, S')$, with associated crossed-product algebras $\mathcal{A}_k$ and $\mathcal{B}_k$ and groupoids $\mathcal{G} = \mathcal{G}(X_\alpha, S)$ and $\mathcal{G}' = \mathcal{G}(X'_{\alpha'}, S')$ a *geometric correspondence* is given by a $(\mathcal{G}, \mathcal{G}')$-space $Z = \operatorname{Spec}(C)$, endowed with a right action of $\mathcal{G}'$ which fulfills the following *étale* condition. Given a space such as $\mathcal{G}'$, that is, a disjoint union of zero-dimensional pro-varieties over k, a right action of $\mathcal{G}'$ on Z is given by a map $g\colon Z \to X'$ and a collection of partial isomorphisms

$$z \in g^{-1}(F(s)) \mapsto z \cdot s \in g^{-1}(E(s)) \tag{4.3}$$

fulfilling the following rules for partial action of the abelian group $\widetilde{S}$:

$$g(z \cdot s) = g(z) \cdot s, \quad z \cdot (ss') = (z \cdot s) \cdot s' \quad \text{on } g^{-1}(F(s) \cap s(F(s'))). \tag{4.4}$$

Here $x \mapsto x \cdot s$ denotes the partial action of $\widetilde{S}$ on X'. One checks that such an action gives to the k-linear space C a structure of right module over $\mathcal{B}_k$. The action of $\mathcal{G}'$ on Z is étale if the corresponding module C is *finite and projective* over $\mathcal{B}_k$.

Given two systems (X_α, S) and $(X'_{\alpha'}, S')$ as above, an *étale correspondence* is therefore a $(\mathcal{G}(X_\alpha, S), \mathcal{G}(X'_{\alpha'}, S'))$-space Z such that the right action of $\mathcal{G}(X'_{\alpha'}, S')$ is étale.

The $\mathbb{Q}$-linear space of (*virtual*) *correspondences*

$$\operatorname{Corr}((X_\alpha, S), (X'_{\alpha'}, S'))$$

is the rational vector space of formal linear combinations $U = \sum_i a_i Z_i$ of étale correspondences Z_i, modulo the relations arising from isomorphisms and equivalences: $Z \coprod Z' \sim Z + Z'$. The composition of correspondences is given by the fiber product over a groupoid. Namely, for three systems (X_α, S), $(X'_{\alpha'}, S')$, $(X''_{\alpha''}, S'')$ joined by correspondences

$$(X_\alpha, S) \leftarrow Z \to (X'_{\alpha'}, S'), \quad (X'_{\alpha'}, S') \leftarrow W \to (X''_{\alpha''}, S''),$$

their composition is given by the rule

$$Z \circ W = Z \times_{\mathcal{G}'} W, \tag{4.5}$$

which is the fiber product over the groupoid $\mathcal{G}' = \mathcal{G}(X'_{\alpha'}, S')$.

Finally, a system (X_α, S) as above is said to be *uniform* if the normalized counting measures μ_α on X_α satisfy $\xi_{\beta,\alpha}\mu_\alpha = \mu_\beta$.

Definition 4.1. The category $\mathcal{E}\mathcal{V}^o(k)_K$ of *algebraic endomotives* with coefficients in a fixed extension K of $\mathbb{Q}$ is the (pseudo)abelian category generated by the following objects and morphisms. The objects are uniform systems $M = (X_\alpha, S)$ of Artin motives over k, as above. The set of morphisms in the category connecting two objects $M = (X_\alpha, S)$ and $M' = (X'_{\alpha'}, S')$ is defined as

$$\mathrm{Hom}(M, M') = \mathrm{Corr}((X_\alpha, S), (X'_{\alpha'}, S')) \otimes_{\mathbb{Q}} K.$$

The category $\mathcal{C}\mathcal{V}^o(k)_K$ of Artin motives embeds as a full subcategory in the category of algebraic endomotives

$$\iota \colon \mathcal{C}\mathcal{V}^o(k)_K \to \mathcal{E}\mathcal{V}^o(k)_K.$$

The functor ι maps an Artin motive $M = X$ to the system (X_α, S) with $X_\alpha = X$ for all α and $S = \{\mathrm{id}\}$.

4.1 Examples of algebraic endomotives

The category of algebraic endomotives is inclusive of a large and general class of examples of noncommutative spaces $\mathcal{A}_k = A \rtimes S$ which are described by semigroup actions on projective systems of Artin motives.

One may consider, for instance a pointed algebraic variety (Y, y_0) over a field k and a countable, unital, abelian semigroup S of *finite* endomorphisms of (Y, y_0), *unramified* over $y_0 \in Y$. Then, there is a system (X_s, S) of Artin motives over k which is constructed from these data. More precisely, for $s \in S$, one sets

$$X_s = \{y \in Y : s(y) = y_0\}. \tag{4.6}$$

For a pair $s, s' \in S$, with $s' = sr$, the connecting map $\xi_{s,s'} \colon X_{sr} \to X_s$ is defined by

$$X_{sr} \ni y \mapsto r(y) \in X_s. \tag{4.7}$$

This is an example of a system indexed by the semigroup S itself, with partial order given by divisibility. One sets $X = \varprojlim_s X_s$.

Since $s(y_0) = y_0$, the base point y_0 defines a component Z_s of X_s for all $s \in S$. The pre-image $\xi_{s,s'}^{-1}(Z_s)$ in $X_{s'}$ is a union of components of $X_{s'}$. This defines a projection e_s onto an open and closed subset X^{e_s} of the projective limit X.

It is easy to see that the semigroup S acts on the projective limit X by partial isomorphisms $\beta_s\colon X \to X^{e_s}$ defined by the property

$$\beta_s\colon X \to X^{e_s}, \quad \xi_{su}(\beta_s(x)) = \xi_u(x) \quad \text{for all } u \in S,\ x \in X. \tag{4.8}$$

The map β_s is well-defined since the set $\{su : u \in S\}$ is cofinal and $\xi_u(x) \in X_{su}$, with $su\xi_u(x) = s(y_0) = y_0$. The image of β_s is in X^{e_s}, since by definition of β_s: $\xi_s(\beta_s(x)) = \xi_1(x) = y_0$. For $x \in X^{e_s}$, we have $\xi_{su}(x) \in X_u$. This shows that β_s defines an isomorphism of X with X^{e_s}, whose inverse map is given by

$$\xi_u(\beta_s^{-1}(x)) = \xi_{su}(x) \quad \text{for all } x \in X^{e_s},\ u \in S. \tag{4.9}$$

The corresponding algebra morphisms ρ_s are then given by

$$\rho_s(f)(x) = f(\beta_s^{-1}(x)) \ \text{ for all } x \in X^{e_s}, \quad \rho_s(f)(x) = 0 \ \text{ for all } x \notin X^{e_s}. \tag{4.10}$$

This class of examples also fulfill an *equidistribution property*, making the uniform normalized counting measures μ_s on X_s compatible with the projective system and inducing a probability measure on the limit X. Namely, one has

$$\xi_{s',s}\mu_s = \mu_{s'} \quad \text{for all } s, s' \in S. \tag{4.11}$$

For a detailed study of a particularly relevant example of an algebraic endomotive we refer to the recent paper [11].

5 Analytic endomotives

In this section we assume that k is a number field. We fix an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$ and we denote by $\bar{k}$ an algebraic closure of $\sigma(k) \subset \mathbb{C}$ in $\mathbb{C}$.

When taking points over $\bar{k}$, algebraic endomotives yield 0-dimensional singular quotient spaces $X(\bar{k})/S$, which can be described by means of locally compact étale groupoids $\mathcal{G}(\bar{k})$ and the associated crossed product C^*-algebras $C(X(\bar{k})) \rtimes S$. This construction gives rise to the category of *analytic endomotives*.

One starts off by considering a uniform system (A_α, S) of Artin motives over k and the algebras

$$A_{\mathbb{C}} = A \otimes_k \mathbb{C} = \varinjlim_\alpha A_\alpha \otimes_k \mathbb{C}, \quad \mathcal{A}_{\mathbb{C}} = \mathcal{A}_k \otimes_k \mathbb{C} = A_{\mathbb{C}} \rtimes S. \tag{5.1}$$

The assignment

$$a \in A \to \hat{a}, \quad \hat{a}(\chi) = \chi(a) \quad \text{for all } \chi \in X = \varprojlim_\alpha X_\alpha \tag{5.2}$$

defines an involutive embedding of algebras $A_{\mathbb{C}} \subset C(X)$. The C^*-completion $C(X)$ of $A_{\mathbb{C}}$ is an abelian AF C^*-algebra. One sets

$$\bar{\mathcal{A}}_{\mathbb{C}} = C(X) \rtimes S.$$

This is the C^*-completion of the algebraic crossed product $A_{\mathbb{C}} \rtimes S$. It is defined by the algebraic relations (4.1) with the involution which is apparent in the formulae (*cf.* [28], [29]).

In the applications that require to work with cyclic (co)homology, it is important to be able to restrict from C^*-algebras such as $\bar{\mathcal{A}}_{\mathbb{C}}$ to canonical dense subalgebras

$$\mathcal{A}_{\mathbb{C}} = C^\infty(X) \rtimes_{\text{alg}} S \subset \bar{\mathcal{A}}_{\mathbb{C}} \tag{5.3}$$

where $C^\infty(X) \subset C(X)$ is the algebra of locally constant functions. It is to this category of smooth algebras (rather than to that of C^*-algebras) that cyclic homology applies.

The following result plays an important role in the theory of endomotives and their applications to examples arising from the study of the thermodynamical properties of certain quantum statistical dynamical systems. We shall refer to the following proposition, in Section 5.1 of this paper for the description of the properties of the "BC-system". The BC-system is a particularly relevant quantum statistical dynamical system which has been the prototype and the motivating example for the introduction of the notion of an endomotive. We refer to [10], § 4.1 for the definition of the notion and the properties of a state on a (unital) involutive algebra.

Proposition 5.1 ([10] Proposition 3.1). 1) *The action* (4.2) *of* G_k *on* $X(\bar{k})$ *defines a canonical action of* G_k *by automorphisms of the* C^**-algebra* $\bar{\mathcal{A}}_{\mathbb{C}} = C(X) \rtimes S$*, preserving globally* $C(X)$ *and such that, for any pure state* φ *of* $C(X)$*,*

$$\alpha\, \varphi(a) = \varphi(\alpha^{-1}(a)) \quad \textit{for all } a \in A,\ \alpha \in G_k. \tag{5.4}$$

2) *When the Artin motives* A_α *are abelian and normal, the subalgebras* $A \subset C(X)$ *and* $\mathcal{A}_k \subset \bar{\mathcal{A}}_{\mathbb{C}}$ *are globally invariant under the action of* G_k *and the states* φ *of* $\bar{\mathcal{A}}_{\mathbb{C}}$ *induced by pure states of* $C(X)$ *fulfill*

$$\begin{aligned} \alpha\varphi(a) &= \varphi(\theta(\alpha)(a)) \quad \textit{for all } a \in \mathcal{A}_k, \\ \theta(\alpha) &= \alpha^{-1} \quad \textit{for all } \alpha \in G_k^{ab} = G_k/[G_k, G_k] \end{aligned} \tag{5.5}$$

On the totally disconnected compact space X, the abelian semigroup S of homeomorphisms acts, producing closed and open subsets $X^s \hookrightarrow X$, $x \mapsto x \cdot s$. The normalized counting measures μ_α on X_α define a *probability measure* on X with the property that the Radon–Nikodym derivatives

$$\frac{ds^*\mu}{d\mu} \tag{5.6}$$

are locally constant functions on X. One lets $\mathcal{G} = X \rtimes S$ be the corresponding étale locally compact groupoid. The crossed product C^*-algebra $C(X(\bar{k})) \rtimes S$ coincides with the C^*-algebra $C^*(\mathcal{G})$ of the groupoid $\mathcal{G}$.

The notion of right (or left) action of $\mathcal{G}$ on a totally disconnected locally compact space $\mathcal{Z}$ is defined as in the algebraic case by (4.3) and (4.4). A right action of $\mathcal{G}$ on $\mathcal{Z}$ gives on the space $C_c(\mathcal{Z})$ of continuous functions with compact support on $\mathcal{Z}$

a structure of right module over $C_c(\mathcal{G})$. When the fibers of the map $g\colon \mathcal{Z} \to X$ are discrete (countable) subsets of $\mathcal{Z}$ one can define on $C_c(\mathcal{Z})$ an inner product with values in $C_c(\mathcal{G})$ by

$$\langle \xi, \eta \rangle (x, s) = \sum_{z \in g^{-1}(x)} \bar{\xi}(z)\, \eta(z \circ s) \tag{5.7}$$

A right action of $\mathcal{G}$ on $\mathcal{Z}$ is *étale* if and only if the fibers of the map g are discrete and the identity is a compact operator in the right C^*-module $\mathcal{E}_{\mathcal{Z}}$ over $C^*(\mathcal{G})$ given by (5.7).

An *étale correspondence* is a $(\mathcal{G}(X_\alpha, S), \mathcal{G}(X'_{\alpha'}, S'))$-space $\mathcal{Z}$ such that the right action of $\mathcal{G}(X'_{\alpha'}, S')$ is étale.

The $\mathbb{Q}$-vector space

$$\mathrm{Corr}((X, S, \mu), (X', S', \mu'))$$

of linear combinations of étale correspondences $\mathcal{Z}$ modulo the equivalence relation $\mathcal{Z} \cup \mathcal{Z}' = \mathcal{Z} + \mathcal{Z}'$ for disjoint unions, defines the space of (virtual) correspondences.

For $M = (X, S, \mu)$, $M' = (X', S', \mu')$, and $M'' = (X'', S'', \mu'')$, the composition of correspondences

$$\mathrm{Corr}(M, M') \times \mathrm{Corr}(M', M'') \to \mathrm{Corr}(M, M''), \quad (Z, W) \mapsto Z \circ W$$

is given following the same rule as for the algebraic case (4.5), that is by the fiber product over the groupoid $\mathcal{G}'$. A correspondence gives rise to a bimodule $\mathcal{M}_{\mathcal{Z}}$ over the algebras $C(X) \rtimes S$ and $C(X') \rtimes S'$ and the composition of correspondences translates into the tensor product of bimodules.

Definition 5.2. The category $C^*\mathcal{V}^o_K$ of *analytic endomotives* is the (pseudo)abelian category generated by objects of the form $M = (X, S, \mu)$ with the properties listed above and morphisms given as follows. For $M = (X, S, \mu)$ and $M' = (X', S', \mu')$ objects in the category, one sets

$$\mathrm{Hom}_{C^*\mathcal{V}^0_K}(M, M') = \mathrm{Corr}(M, M') \otimes_{\mathbb{Q}} K. \tag{5.8}$$

The following result establishes a precise relation between the categories of Artin motives and (noncommutative) endomotives.

Theorem 5.3 ([10], Theorem 3.13). *The categories of Artin motives and algebraic and analytic endomotives are related as follows.*

(1) *The map $\mathcal{G} \mapsto \mathcal{G}(\bar{k})$ determines a tensor functor*

$$\mathcal{F} : \mathcal{E}\mathcal{V}^o(k)_K \to C^*\mathcal{V}^o_K, \quad \mathcal{F}(X_\alpha, S) = (X(\bar{k}), S, \mu)$$

from algebraic to analytic endomotives.

(2) *The Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ acts by natural transformations of $\mathcal{F}$.*

(3) *The category $\mathcal{C}\mathcal{V}^o(k)_K$ of Artin motives embeds as a full subcategory of $\mathcal{E}\mathcal{V}^o(k)_K$.*

(4) *The composite functor*

$$c \circ \mathcal{F} : \mathcal{EV}^o(k)_K \to \mathcal{KK} \otimes K \tag{5.9}$$

maps the full subcategory $\mathcal{CV}^o(k)_K$ of Artin motives faithfully to the category $\mathcal{KK}_{G_k} \otimes K$ of G_k-equivariant KK-theory with coefficients in K.

Given two Artin motives $X = \operatorname{Spec}(A)$ and $X' = \operatorname{Spec}(B)$ and a component $Z = \operatorname{Spec}(C)$ of the cartesian product $X \times X'$, the two projections turn C into a (A, B)-bimodule $c(Z)$. If $U = \sum a_i \chi_{Z_i} \in \operatorname{Hom}_{\mathcal{CV}^o(k)_K}(X, X')$, $c(U) = \sum a_i c(Z_i)$ defines a sum of bimodules in $\mathcal{KK} \otimes K$. The composition of correspondences in $\mathcal{CV}^o(k)_K$ translates into the tensor product of bimodules in $\mathcal{KK}_{G_k} \otimes K$

$$c(U) \otimes_B c(L) \simeq c(U \circ L).$$

One composes the functor c with the natural functor $A \to A_{\mathbb{C}}$ which associates to a (A, B)-bimodule $\mathcal{E}$ the $(A_{\mathbb{C}}, B_{\mathbb{C}})$-bimodule $\mathcal{E}_{\mathbb{C}}$. The resulting functor

$$c \circ \mathcal{F} \circ \iota \colon \mathcal{CV}^o(k)_K \to \mathcal{KK}_{G_k} \otimes K$$

is faithful since a correspondence such as U is uniquely determined by the corresponding map of K-theory $K_0(A_{\mathbb{C}}) \otimes K \to K_0(B_{\mathbb{C}}) \otimes K$.

5.1 The endomotive of the BC system

The prototype example of the data which define an analytic endomotive is the system introduced by Bost and Connes in [6] (*cf.* also [11] for a very recent development on the study of this system). The evolution of this C^*-dynamical system encodes in its group of symmetries the arithmetic of the maximal abelian extension of $k = \mathbb{Q}$.

This quantum statistical dynamical system is described by the datum given by a noncommutative C^*-algebra of *observables* $\bar{\mathcal{A}}_{\mathbb{C}} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^\times$ and by the *time evolution* which is assigned in terms of a one-parameter family of automorphisms σ_t of the algebra. The action of the (multiplicative) semigroup $S = \mathbb{N}^\times$ on the commutative algebra $C^*(\mathbb{Q}/\mathbb{Z}) \simeq C(\hat{\mathbb{Z}})$ is defined by

$$\alpha_n(f)(\rho) = \begin{cases} f(n^{-1}\rho) & \text{if } \rho \in n\hat{\mathbb{Z}}, \\ 0 & \text{otherwise}, \end{cases} \qquad \rho \in \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}.$$

For the definition of the associate endomotive, one considers the projective system $\{X_n\}_{n\in\mathbb{N}}$ of zero-dimensional algebraic varieties $X_n = \operatorname{Spec}(A_n)$, where $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$ is the group ring of the abelian group $\mathbb{Z}/n\mathbb{Z}$. The inductive limit $A = \varinjlim_n A_n = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ is the group ring of $\mathbb{Q}/\mathbb{Z}$. The endomorphism $\rho_n \colon A \to A$ associated to an element $n \in S$ is given on the canonical basis $e_r \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$, $r \in \mathbb{Q}/\mathbb{Z}$, by

$$\rho_n(e_r) = \frac{1}{n} \sum_{ns=r} e_s. \tag{5.10}$$

The Artin motives X_n are normal and abelian, so that Proposition 5.1 applies. The action of the Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $X_n = \mathrm{Spec}(A_n)$ is obtained by composing a character $\chi\colon A_n \to \bar{\mathbb{Q}}$ with the action of an element $g \in G_{\mathbb{Q}}$. Since χ is determined by the n-th root of unity $\chi(e_{1/n})$, this implies that the action of $G_{\mathbb{Q}}$ factorizes through the cyclotomic action and coincides with the *symmetry group* of the BC-system. The subalgebra $\mathcal{A}_{\mathbb{Q}} \subset \bar{\mathcal{A}}_{\mathbb{C}} = C(X) \rtimes S$ coincides with the rational subalgebra defined in [6].

There is an interesting description of this system in terms of a pointed algebraic variety (Y, y_0) (*cf.* Section 4.1) on which the abelian semigroup S acts by finite endomorphisms. One considers the pointed affine group scheme $(\mathbb{G}_m, 1)$ (the multiplicative group) and lets S be the semigroup of non-zero endomorphisms of $\mathbb{G}_m$. These endomorphisms correspond to maps of the form $u \mapsto u^n$, for some $n \in \mathbb{N}$. Then, the general construction outlined in Section 4.1 determines on $(\mathbb{G}_m(\mathbb{Q}), 1)$ the BC system.

One considers the semigroup $S = \mathbb{N}^{\times}$ acting on $\mathbb{G}_m(\mathbb{Q})$ as specified above. It follows from the definition (4.6) that $X_n = \mathrm{Spec}(A_n)$ where

$$A_n = \mathbb{Q}[u_n^{\pm 1}]/(u_n^n - 1).$$

For $n|m$ the connecting morphism $\xi_{m,n}\colon X_m \to X_n$ is defined by the algebra homomorphism $A_n \to A_m$, $u_n^{\pm 1} \mapsto u_m^{\pm a}$ with $a = m/n$. Thus, one obtains an isomorphism of $\mathbb{Q}$-algebras

$$\iota\colon A = \varinjlim_n A_n \xrightarrow{\sim} \mathbb{Q}[\mathbb{Q}/\mathbb{Z}], \quad \iota(u_n) = e_{\frac{1}{n}}. \tag{5.11}$$

The partial isomorphisms $\rho_n\colon \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ of the group ring as described by the formula (5.10) correspond under the isomorphism ι, to those given by (4.8) on $X = \varprojlim_n X_n$. One identifies X with its space of characters $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \to \bar{\mathbb{Q}}$. Then, the projection $\xi_m(x)$ is given by the restriction of (the character associated to) $x \in X$ to the subalgebra A_m. The projection of the composite of the endomorphism ρ_n of (5.10) with $x \in X$ is given by

$$x(\rho_n(e_r)) = \frac{1}{n} \sum_{ns=r} x(e_s).$$

This projection is non-zero if and only if the restriction $x|_{A_n}$ is the trivial character, that is if and only if $\xi_n(x) = 1$. Moreover, in that case one has

$$x(\rho_n(e_r)) = x(e_s) \quad \text{for all } s,\ ns = r,$$

and in particular

$$x(\rho_n(e_{\frac{1}{k}})) = x(e_{\frac{1}{nk}}). \tag{5.12}$$

For $k|m$ the inclusion of algebraic spaces $X_k \subset X_m$ is given at the algebra level by the surjective homomorphism

$$j_{k,m}\colon A_m \to A_k, \quad j_{k,m}(u_m) = u_k.$$

Thus, one can rewrite (5.12) as

$$x \circ \rho_n \circ j_{k,nk} = x|_{A_{nk}}. \tag{5.13}$$

This means that

$$\xi_{nk}(x) = \xi_k(x \circ \rho_n).$$

By using the formula (4.9), one obtains the desired equality of the ρ's of (5.10) and (4.10).

This construction continues to make sense for the affine algebraic variety $\mathbb{G}_m(k)$ for any field k, including the case of a field of positive characteristic. In this case one obtains new systems, different from the BC system.

6 Applications: the geometry of the space of adèles classes

The functor

$$\mathcal{F} : \mathcal{E}\mathcal{V}^o(k)_K \to C^*\mathcal{V}^o_K$$

which connects the categories of algebraic and analytic endomotives establishes a significant bridge between the commutative world of Artin motives and that of noncommutative geometry. When one moves from commutative to noncommutative algebras, important new tools of thermodynamical nature become available. One of the most relevant techniques (for number-theoretical applications) is supplied by the theory of Tomita and Takesaki for von Neumann algebras ([37]) which associates to a suitable state φ (*i.e.* a faithful weight) on a von Neumann algebra M, a one-parameter group of automorphisms of M (*i.e.* the modular automorphism group)

$$\sigma_t^\varphi : \mathbb{R} \to \mathrm{Aut}(M), \quad \sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}.$$

Δ_φ is the modular operator which acts on the completion $L^2(M, \varphi)$ of $\{x \in M : \varphi(x^*x) < \infty\}$, for the scalar product $\langle x, y\rangle = \varphi(y^*x)$.

This general theory applies in particular to the unital involutive algebras $\mathcal{A} = C^\infty(X) \rtimes_{\mathrm{alg}} S$ of (5.3) and to the related C^*-algebras $\bar{\mathcal{A}}_{\mathbb{C}}$ which are naturally associated to an endomotive.

A remarkable result proved by Connes in the theory and classification of factors ([7]) states that, modulo inner automorphisms of M, the one-parameter family σ_t^φ is *independent* of the choice of the state φ. This way, one obtains a *canonically defined* one parameter group of automorphism classes

$$\delta : \mathbb{R} \to \mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M).$$

In turn, this result implies that the crossed product dual algebra

$$\hat{M} = M \rtimes_{\sigma_t^\varphi} \mathbb{R}$$

and the dual scaling action

$$\theta_\lambda : \mathbb{R}_+^* \to \mathrm{Aut}(\widehat{M}) \tag{6.1}$$

are *independent* of the choice of (the weight) φ.

When these results are applied to the analytic endomotive $\mathcal{F}(X_\alpha, S)$ associated to an algebraic endomotive $M = (X_\alpha, S)$, the above dual representation of $\mathbb{R}_+^*$ combines with the representation of the absolute Galois group G_k. In the particular case of the endomotive associated to the BC-system (*cf.* Section 5.1), the resulting representation of $G_{\mathbb{Q}} \times \mathbb{R}_+^*$ on the cyclic homology HC_0 of a suitable $\mathbb{Q}(\Lambda)$-module $D(\mathcal{A}, \varphi)$ associated to the thermodynamical dynamics of the system $(\mathcal{A}, \sigma_t^\varphi)$ determines the spectral realization of the zeroes of the Riemann zeta-function and of the Artin L-functions for abelian characters of G_k (*cf.* [10], Theorem 4.16).

The action of the group $W = G_{\mathbb{Q}} \times \mathbb{R}_+^*$ on the cyclic homology $HC_0(D(\mathcal{A}, \varphi))$ of the noncommutative motive $D(\mathcal{A}, \varphi)$ is analogous to the action of the Weil group on the étale cohomology of an algebraic variety. In particular, the action of $\mathbb{R}_+^*$ is the 'characteristic zero' analog of the action of the (geometric) Frobenius on étale cohomology. This construction determines a functor

$$\omega : \mathcal{EV}^o(k)_K \to \mathrm{Rep}_{\mathbb{C}}(W)$$

from the category of endomotives to the category of (infinite-dimensional) representations of the group W.

The analogy with the Tannakian formalism of classical motive theory is striking. It is also important to underline the fact that the whole thermodynamical construction is non-trivial and relevant for number-theoretic applications only because of the particular nature of the factor M (type III_1) associated to the original datum $(\mathcal{A}, \varphi)$ of the BC-system.

It is tempting to compare the original choice of the state φ (weight) on the algebra $\mathcal{A}$ which singles out (via the Gelfand–Naimark–Segal construction) the factor M defined as the weak closure of the action of $\bar{\mathcal{A}} = C(X) \rtimes S$ in the Hilbert space $\mathcal{H}_\varphi = L^2(M, \varphi)$, with the assignment of a factor $(X, p, m)^r$ ($r \in \mathbb{Z}$) on a pure motive $M = (X, p, m)$, *cf.* (2.8). In classical motive theory, one knows that the assignment of a $\mathbb{Z}$-grading is canonical only for homological equivalence or under the assumption of the Standard Conjecture of Künneth type. In fact, the definition of a weight structure depends upon the definition of a complete system of orthogonal central idempotents π_X^i.

Passing from the factor M to the canonical dual representation (6.1) carries also the advantage to work in a setting where projectors are classified by their real dimension ($\widehat{M}$ is of type II_1), namely in a noncommutative framework of continuous geometry which generalizes and yet still retains some relevant properties of the algebraic correspondences (*i.e.* degree or dimension).

The process of dualization is in fact subsequent to a thermodynamical "cooling procedure" in order to work with a system whose algebra approaches and becomes in the limit, a commutative algebra (*i.e.* I_∞). Finally, one has also to implement a further step in which one filters (*i.e.* "distils") the relevant noncommutative motive $D(\mathcal{A}, \varphi)$ within the derived framework of cyclic modules (*cf.* Section 3.1). This procedure is somewhat reminiscent of the construction of the vanishing cohomology in algebraic geometry (*cf.* [24]).

When the algebra of the BC-system gets replaced by the noncommutative algebra of coordinates $\mathcal{A} = \mathcal{S}(\mathbb{A}_k) \rtimes k^*$ of the adèle class space $X_k = \mathbb{A}_k/k^*$ of a number field k, the cooling procedure is described by a restriction morphism of (rapidly decaying) functions on X_k to functions on the "cooled down" subspace C_k of idèle classes (*cf.* [10], Section 5). In this context, the representation of C_k on the cyclic homology $HC_0(\mathcal{H}^1_{k,\mathbb{C}})$ of a suitable noncommutative motive $\mathcal{H}^1_{k,\mathbb{C}}$ produces the spectral realization of the zeroes of Hecke L-functions (*cf.* *op.cit*, Theorem 5.6).

The whole construction describes also a natural way to associate to a noncommutative space a canonical set of "classical points" which represents the analogue in characteristic zero, of the geometric points $C(\overline{\mathbb{F}}_q)$ of a smooth, projective curve $C_{/\overline{\mathbb{F}}_q}$.

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Renormalisation of non-commutative field theories

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Abstract. The first renormalisable quantum field theories on non-commutative space have been found recently. We review this rapidly growing subject.

Contents

*This review follows lectures given by V. R. at the workshop "Renormalisation et théories de Galois", Luminy, March 2006.

1 Introduction

General relativity and ordinary differential geometry should be replaced by non-commutative geometry at some point between the currently accessible energies of about 1 – 10 Tev (after starting the Large Hadron Collider (LHC) at CERN) and the Planck scale, which is 10^{15} times higher, where space-time and gravity should be quantized.

This could occur either at the Planck scale or below. Quantum field theory on a non-commutative space-time (NCQF) could very well be an intermediate theory relevant for physics at energies between the LHC and the Planck scale. It certainly looks intermediate in structure between ordinary quantum field theory on commutative $\mathbb{R}^4$ and string theory, the current leading candidate for a more fundamental theory including quantized gravity. NCQFT in fact appears as an effective model for certain limits of string theory [1], [2].

In joint work with R. Gurau, J. Magnen and F. Vignes-Tourneret [3], using direct space methods, we provided recently a new proof that the Grosse–Wulkenhaar scalar Φ^4_4-theory on the Moyal space $\mathbb{R}^4$ is renormalisable to all orders in perturbation theory.

The Grosse–Wulkenhaar breakthrough [4], [5] was to realize that the right propagator in non-commutative field theory is not the ordinary commutative propagator, but has to be modified to obey Langmann–Szabo duality [6], [5]. Grosse and Wulkenhaar were able to compute the corresponding propagator in the so called "matrix base" which transforms the Moyal product into a matrix product. This is a real *tour de force*! They use this representation to prove perturbative renormalisability of the theory up to some estimates which were finally proven in [7].

Our direct space method builds upon the previous works of Filk and Chepelev–Roiban [8], [9]. These works however remained inconclusive [10], since these authors used the right interaction but not the right propagator, hence the problem of ultraviolet/infrared mixing prevented them from obtaining a finite renormalised perturbation series.

We also extend the Grosse–Wulkenhaar results to more general models with covariant derivatives in a fixed magnetic field [11]. Our proof relies on a multiscale analysis analogous to [7] but in direct space.

Non-commutative field theories (for a general review see [12]) deserve a thorough and systematic investigation, not only because they may be relevant for physics beyond the standard model, but also (although this is often less emphasized) because they can describe effective physics in our ordinary standard world but with non-local interactions.

In this case there is an interesting reversal of the initial Grosse–Wulkenhaar problematic. In the Φ^4_4-theory on the Moyal space $\mathbb{R}^4$, the vertex is sort of God-given by the Moyal structure, and it is LS invariant. The challenge was to overcome uv/ir mixing and to find the right propagator which makes the theory renormalisable. This propagator turned out to have LS duality. The harmonic potential introduced by Grosse and Wulkenhaar can be interpreted as a piece of covariant derivatives in a constant magnetic field.

Now to explain the (fractional) quantum Hall effect, which is a bulk effect whose understanding requires electron interactions, we can almost invert this logic. The propagator is known since it corresponds to non-relativistic electrons in two dimensions in a constant magnetic field. It has LS duality. But the interaction is unclear, and cannot be local since at strong magnetic field the spins should align with the magnetic field, hence by Pauli principle local interactions among electrons in the first Landau level should vanish.

We can argue that among all possible non-local interactions, a few renormalisation group steps should select the only ones which form a renormalisable theory with the corresponding propagator. In the commutative case (i.e. zero magnetic field) local interactions such as those of the Hubbard model are just renormalisable in any dimension because of the extended nature of the Fermi-surface singularity. Since the non-commutative electron propagator (i.e. in non-zero magnetic field) looks very similar to the Grosse–Wulkenhaar propagator (it is in fact a generalization of the Langmann–Szabo–Zarembo propagator) we can conjecture that the renormalisable interaction corresponding to this propagator should be given by a Moyal product. That is why we hope that non-commutative field theory is the correct framework for a microscopic *ab initio* understanding of the fractional quantum Hall effect which is currently lacking.

Even for regular commutative field theory such as non-Abelian gauge theory, the strong coupling or non-perturbative regimes may be studied fruitfully through their non-commutative (i.e. non-local) counterparts. This point of view is forcefully suggested in [2], where a mapping is proposed between ordinary and non-commutative gauge fields which do not preserve the gauge groups but preserve the gauge equivalent classes. We can at least remark that the effective physics of confinement should be governed by a non-local interaction, as is the case in effective strings or bags models.

In other words we propose to base physics upon the renormalisability principle, more than any other axiom. Renormalisability means genericity; only renormalisable interactions survive a few RG steps, hence only them should be used to describe generic effective physics of any kind. The search for renormalisabilty could be the powerful principle on which to orient ourselves in the jungle of all possible non-local interactions.

Renormalisability has also attracted considerable interest in the recent years as a pure mathematical structure. The work of Kreimer and Connes [13], [14], [15] recasts the recursive BPHZ forest formula of perturbative renormalisation in a nice Hopf algebra structure. The renormalisation group ambiguity reminds mathematicians of the Galois group ambiguity for roots of algebraic equations. Finding new renormalisable theories may therefore be important for the future of pure mathematics as well as for physics. That was forcefully argued during the Luminy workshop “Renormalisation and Galois Theory”. Main open conjectures in pure mathematics such as the Riemann hypothesis [16], [17] or the Jacobian conjecture [18] may benefit from the quantum field theory and renormalisation group approach.

Considering that most of the Connes–Kreimer works use dimensional regularization and the minimal dimensional renormalisation scheme, it is interesting to develop the parametric representation which generalize Schwinger's parametric representation of Feynman amplitudes to the non-commutative context. It involves hyperbolic generalizations of the ordinary topological polynomials, which mathematicians call Kirchoff polynomials, and physicist call Symanzik polynomials in the quantum field theory context [19]. We plan also to work out the corresponding regularization and minimal dimensional renormalisation scheme and to recast it in a Hopf algebra structure. The corresponding structures seem richer than in ordinary field theory since they involve ribbon graphs and invariants which contain information about the genus of the surface on which these graphs live.

A critical goal to enlarge the class of renormalisable non-commutative field theories and to attack the Quantum Hall effect problem is to extend the results of Grosse–Wulkenhaar to Fermionic theories. The simplest theory, the two-dimensional Gross–Neveu model can be shown renormalisable to all orders in their Langmann–Szabo covariant versions, using either the matrix basis [20] or the direct space version developed here [21]. However the x-space version seems the most promising for a complete non-perturbative construction, using Pauli's principle to control the apparent (fake) divergences of perturbation theory.

In the case of ϕ^4_4, recall that although the commutative version is until now fatally flawed due to the famous Landau ghost, there is hope that the non-commutative field theory treated at the perturbative level in this paper may also exist at the constructive level. Indeed a non-trivial fixed point of the renormalisation group develops at high energy, where the Grosse–Wulkenhaar parameter Ω tends to 1, so that Langmann–Szabo duality become exact, and the beta function vanishes. This scenario has been checked explicitly to all orders of perturbation theory [22], [23], [24]. This was done using the matrix version of the theory; again an x-space version of renormalisation might be better for a future rigorous non-perturbative investigation of this fixed point and a full constructive version of the model.

Finally let us conclude this short introduction by reminding that a very important and difficult goal is to also extend the Grosse–Wulkenhaar breakthrough to gauge theories.

1.1 The quantum Hall effect

One considers free electrons: $H_0 = \frac{1}{2m}(\boldsymbol{p} + e\boldsymbol{A})^2 = \frac{\pi^2}{2m}$ where $\boldsymbol{p} = m\dot{r} - eA$ is the canonical conjugate of r.

The moment and position p and r have commutators

$$[p_i, p_j] = 0, \quad [r_i, r_j] = 0, \quad [p_i, r_j] = \imath\hbar\delta_{ij}. \tag{1.1}$$

The moments $\boldsymbol{\pi} = m\dot{r} = p + eA$ have commutators

$$[\pi_i, \pi_j] = -\imath\hbar\varepsilon_{ij}eB, \quad [r_i, r_j] = 0, \quad [\pi_i, r_j] = \imath\hbar\delta_{ij}. \tag{1.2}$$

One can also introduce coordinates R_x, R_y corresponding to the centers of the classical trajectories

$$R_x = x - \frac{1}{eB}\pi_y, \quad R_y = y + \frac{1}{eB}\pi_x \tag{1.3}$$

which *do not commute*:

$$[R_i, R_j] = \imath\hbar\varepsilon_{ij}\frac{1}{eB}, \quad [\pi_i, R_j] = 0. \tag{1.4}$$

This means that there exist Heisenberg-like relations between quantum positions.

1.2 String theory in background field

One considers the string action in a generalized background

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma (g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu - 2\pi i\alpha' B_{\mu\nu}\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu) \tag{1.5}$$

$$= \frac{1}{4\pi\alpha'}\int_\Sigma g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu - \frac{i}{2}\int_{\partial\Sigma} B_{\mu\nu}X^\mu \partial_t X^\nu, \tag{1.6}$$

where Σ is the string worldsheet, ∂_t is a tangential derivative along the worldsheet boundary $\partial\Sigma$ and $B_{\mu\nu}$ is an antisymmetric background tensor. The equations of motion determine the boundary conditions:

$$g_{\mu\nu}\partial_n X^\mu + 2\pi i\alpha' B_{\mu\nu}\partial_t X^\mu|_{\partial\Sigma} = 0. \tag{1.7}$$

Boundary conditions for coordinates can be Neumann ($B \to 0$) or Dirichlet ($g \to 0$, corresponding to branes).

After conformal mapping of the string worldsheet onto the upper half-plane, the string propagator in background field is

$$\begin{aligned} < X^\mu(z)X^\nu(z') > = -\alpha'\Big[& g^{\mu\nu}(\log|z-z'|\log|z-\bar{z}'|) \\ & + G^{\mu\nu}\log|z-\bar{z}'|^2 + \theta^{\mu\nu}\log\frac{|z-\bar{z}'|}{|\bar{z}-z'|} + \text{const.}\Big] \end{aligned} \tag{1.8}$$

for some constant symmetric and antisymmetric tensors G and θ.

Evaluated at boundary points on the worldsheet, this propagator is

$$< X^\mu(\tau)X^\nu(\tau') > \ = -\alpha' G^{\mu\nu}\log(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\varepsilon(\tau-\tau')\ , \tag{1.9}$$

where the θ term simply comes from the discontinuity of the logarithm across its cut. Interpreting τ as time, one finds

$$[X^\mu, X^\nu] = i\theta^{\mu\nu}, \tag{1.10}$$

which means that string coordinates lie in a non-commutative Moyal space with parameter θ.

There is an equivalent argument inspired by M-theory: a rotation sandwiched between two T dualities generates the same constant commutator for string coordinates.

2 Non-commutative field theory

2.1 Field theory on Moyal space

The recent progresses concerning the renormalisation of non-commutative field theory have been obtained on a very simple non-commutative space namely the Moyal space. From the point of view of quantum field theory, it is certainly the most studied space. Let us start with its precise definition.

2.1.1 The Moyal space $\mathbb{R}^D_\theta$. Let us define $E = \left\{x^\mu,\ \mu \in [\![1, D]\!]\right\}$ and $\mathbb{C}\langle E\rangle$ the free algebra generated by E. Let Θ a $D \times D$ non-degenerate skew-symmetric matrix (which requires D even) and I the ideal of $\mathbb{C}\langle E\rangle$ generated by the elements $x^\mu x^\nu - x^\nu x^\mu - \imath\Theta^{\mu\nu}$. The Moyal algebra $\mathcal{A}_\Theta$ is the quotient $\mathbb{C}\langle E\rangle/I$. Each element in $\mathcal{A}_\Theta$ is a formal power series in the x^μ's for which the relation $[x^\mu, x^\nu] = \imath\Theta^{\mu\nu}$ holds.

Usually, one puts the matrix Θ into its canonical form:

$$\Theta = \begin{pmatrix} \begin{matrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{matrix} & & (0) \\ & \ddots & \\ (0) & & \begin{matrix} 0 & \theta_{D/2} \\ -\theta_{D/2} & 0 \end{matrix} \end{pmatrix}. \tag{2.1}$$

Sometimes one even set $\theta = \theta_1 = \cdots = \theta_{D/2}$. The preceding algebraic definition whereas short and precise may be too abstract to perform real computations. One then needs a more analytical definition. A representation of the algebra $\mathcal{A}_\Theta$ is given by some set of functions on $\mathbb{R}^d$ equipped with a non-commutative product: the *Groenwald–Moyal* product. What follows is based on [25].

The algebra $\mathcal{A}_\Theta$. The Moyal algebra $\mathcal{A}_\Theta$ is the linear space of smooth and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^D)$ equipped with the non-commutative product defined by: for all $f, g \in \mathcal{S}_D \stackrel{\text{def}}{=} \mathcal{S}(\mathbb{R}^D)$,

$$\begin{aligned}(f \star_\Theta g)(x) &= \int_{\mathbb{R}^D} \frac{d^D k}{(2\pi)^D} d^D y\, f(x + \tfrac{1}{2}\Theta \cdot k) g(x + y) e^{\imath k\cdot y} &(2.2)\\ &= \frac{1}{\pi^D\,|\det\Theta|}\int_{\mathbb{R}^D} d^D y d^D z\, f(x + y) g(x + z) e^{-2\imath y\Theta^{-1}z}. &(2.3)\end{aligned}$$

This algebra may be considered as the "functions on the Moyal space $\mathbb{R}^D_\theta$". In the following we will write $f \star g$ instead of $f \star_\Theta g$ and use that for all $f, g \in \mathcal{S}_D$ we have

$$(\mathcal{F} f)(x) = \int f(t) e^{-\imath t x} dt \tag{2.4}$$

for the Fourier transform and

$$(f \diamond g)(x) = \int f(x-t) g(t) e^{2\imath x \Theta^{-1} t} dt \tag{2.5}$$

for the twisted convolution. As on $\mathbb{R}^D$, the Fourier transform exchanges product and convolution:

$$\mathcal{F}(f \star g) = \mathcal{F}(f) \diamond \mathcal{F}(g), \tag{2.6}$$

$$\mathcal{F}(f \diamond g) = \mathcal{F}(f) \star \mathcal{F}(g). \tag{2.7}$$

One also shows that the Moyal product and the twisted convolution are *associative*:

$$((f \diamond g) \diamond h)(x) = \int f(x-t-s) g(s) h(t) e^{2\imath (x\Theta^{-1}t + (x-t)\Theta^{-1}s)} ds\, dt \tag{2.8}$$

$$= \int f(u-v) g(v-t) h(t) e^{2\imath (x\Theta^{-1}v - t\Theta^{-1}v)} dt\, dv$$

$$= (f \diamond (g \diamond h))(x). \tag{2.9}$$

Using (2.7), we show the associativity of the $\star$-product. The complex conjugation is *involutive* in $\mathcal{A}_\Theta$:

$$\overline{f \star_\Theta g} = \bar{g} \star_\Theta \bar{f}. \tag{2.10}$$

One also has

$$f \star_\Theta g = g \star_{-\Theta} f. \tag{2.11}$$

Proposition 2.1 (Trace). *For all $f, g \in \mathcal{S}_D$,*

$$\int dx\, (f \star g)(x) = \int dx\, f(x) g(x) = \int dx\, (g \star f)(x). \tag{2.12}$$

Proof.

$$\int dx\, (f \star g)(x) = \mathcal{F}(f \star g)(0) = (\mathcal{F} f \diamond \mathcal{F} g)(0) \tag{2.13}$$

$$= \int \mathcal{F} f(-t) \mathcal{F} g(t) dt = (\mathcal{F} f * \mathcal{F} g)(0) = \mathcal{F}(fg)(0)$$

$$= \int f(x) g(x) dx,$$

where $*$ is the ordinary convolution. □

In the following sections, we will need Lemma 2.2 to compute the interaction terms for the Φ^4_4 and Gross–Neveu models. We write $x \wedge y \stackrel{\text{def}}{=} 2x\Theta^{-1}y$.

Lemma 2.2. *For all* $j \in [\![1, 2n+1]\!]$, *let* $f_j \in \mathcal{A}_\Theta$. *Then*

$$\begin{aligned}
&(f_1 \star_\Theta \cdots \star_\Theta f_{2n})(x) \\
&\quad = \frac{1}{\pi^{2D} \det^2 \Theta} \int \prod_{j=1}^{2n} dx_j\, f_j(x_j)\, e^{-\imath x \wedge \sum_{i=1}^{2n} (-1)^{i+1} x_i}\, e^{-\imath \varphi_{2n}},
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
&(f_1 \star_\Theta \cdots \star_\Theta f_{2n+1})(x) \\
&\quad = \frac{1}{\pi^{D} \det \Theta} \int \prod_{j=1}^{2n+1} dx_j\, f_j(x_j)\, \delta\Big(x - \sum_{i=1}^{2n+1} (-1)^{i+1} x_i\Big)\, e^{-\imath \varphi_{2n+1}},
\end{aligned} \tag{2.15}$$

$$\varphi_p = \sum_{i<j=1}^{p} (-1)^{i+j+1} x_i \wedge x_j \quad \textit{for all } p \in \mathbb{N}. \tag{2.16}$$

Corollary 2.3. *For all* $j \in [\![1, 2n+1]\!]$, *let* $f_j \in \mathcal{A}_\Theta$. *Then*

$$\begin{aligned}
&\int dx\ (f_1 \star_\Theta \cdots \star_\Theta f_{2n})(x) \\
&\quad = \frac{1}{\pi^{D} \det \Theta} \int \prod_{j=1}^{2n} dx_j\, f_j(x_j)\ \delta\Big(\sum_{i=1}^{2n} (-1)^{i+1} x_i\Big) e^{-\imath \varphi_{2n}},
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&\int dx\ (f_1 \star_\Theta \cdots \star_\Theta f_{2n+1})(x) \\
&\quad = \frac{1}{\pi^{D} \det \Theta} \int \prod_{j=1}^{2n+1} dx_j\, f_j(x_j)\, e^{-\imath \varphi_{2n+1}},
\end{aligned} \tag{2.18}$$

$$\varphi_p = \sum_{i<j=1}^{p} (-1)^{i+j+1} x_i \wedge x_j \quad \textit{for all } p \in \mathbb{N}. \tag{2.19}$$

The cyclicity of the product, inherited from Proposition 2.1, implies that for all $f, g, h \in \mathcal{S}_D$,

$$\langle f \star g, h \rangle = \langle f, g \star h \rangle = \langle g, h \star f \rangle, \tag{2.20}$$

and allows to extend the Moyal algebra by duality into an algebra of tempered distributions.

Extension by duality. Let us first consider the product of a tempered distribution with a Schwartz-class function. Let $T \in \mathcal{S}'_D$ and $h \in \mathcal{S}_D$. We define $\langle T, h \rangle \stackrel{\text{def}}{=} T(h)$ and $\langle T^*, h \rangle = \overline{\langle T, \overline{h} \rangle}$.

Definition 2.1. Let $T \in \mathcal{S}'_D$, $f, h \in \mathcal{S}_D$, we define $T \star f$ and $f \star T$ by

$$\langle T \star f, h\rangle = \langle T, f \star h\rangle, \tag{2.21}$$
$$\langle f \star T, h\rangle = \langle T, h \star f\rangle. \tag{2.22}$$

For example, the identity $\mathbb{1}$ as an element of $\mathcal{S}'_D$ is the unity for the $\star$-product: for all $f, h \in \mathcal{S}_D$,

$$\begin{aligned}\langle \mathbb{1} \star f, h\rangle &= \langle \mathbb{1}, f \star h\rangle \\ &= \int (f \star h)(x)dx = \int f(x)h(x)dx \\ &= \langle f, h\rangle.\end{aligned} \tag{2.23}$$

We are now ready to define the linear space $\mathcal{M}$ as the intersection of two sub-spaces $\mathcal{M}_L$ and $\mathcal{M}_R$ of $\mathcal{S}'_D$.

Definition 2.2 (Multiplier algebra).

$$\mathcal{M}_L = \{S \in \mathcal{S}'_D : S \star f \in \mathcal{S}_D \ \text{ for all} f \in \mathcal{S}_D\}, \tag{2.24}$$
$$\mathcal{M}_R = \{R \in \mathcal{S}'_D : f \star R \in \mathcal{S}_D \ \text{ for all} f \in \mathcal{S}_D\}, \tag{2.25}$$
$$\mathcal{M} = \mathcal{M}_L \cap \mathcal{M}_R. \tag{2.26}$$

One can show that $\mathcal{M}$ is an associative $*$-algebra. It contains, among others, the identity, the polynomials, the δ-distribution and its derivatives. Then the relation

$$[x^\mu, x^\nu] = \imath\Theta^{\mu\nu}, \tag{2.27}$$

often given as a definition of the Moyal space, holds in $\mathcal{M}$ (but not in $\mathcal{A}_\Theta$).

2.1.2 The ϕ^4-theory on $\mathbb{R}^4_\theta$. The simplest non-commutative model one may consider is the ϕ^4-theory on the four-dimensional Moyal space. Its Lagrangian is the usual (commutative) one where the pointwise product is replaced by the Moyal one:

$$S[\phi] = \int d^4x\Big(-\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}m^2\,\phi \star \phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\Big)(x). \tag{2.28}$$

Thanks to the formula (2.3), this action can be explicitly computed. The interaction part is given by Corollary 2.3:

$$\int dx\,\phi^{\star 4}(x) = \int \prod_{i=1}^{4} dx_i\,\phi(x_i)\,\delta(x_1 - x_2 + x_3 - x_4)e^{\imath\varphi}, \tag{2.29}$$
$$\varphi = \sum_{i<j=1}^{4} (-1)^{i+j+1} x_i \wedge x_j\,.$$

The main characteristic of the Moyal product is its non-locality. But its non-commutativity implies that the vertex of the model (2.28) is only invariant under cyclic permutation of the fields. This restricted invariance incites to represent the associated Feynman graphs with ribbon graphs. One can then make a clear distinction between planar and non-planar graphs. This will be detailed in Section 3.

Thanks to the δ-function in (2.29), the oscillation may be written in different ways:

$$\delta(x_1 - x_2 + x_3 - x_4)e^{\imath\varphi} = \delta(x_1 - x_2 + x_3 - x_4)e^{\imath x_1\wedge x_2 + \imath x_3\wedge x_4} \tag{2.30a}$$

$$= \delta(x_1 - x_2 + x_3 - x_4)e^{\imath x_4\wedge x_1 + \imath x_2\wedge x_3} \tag{2.30b}$$

$$= \delta(x_1 - x_2 + x_3 - x_4)\exp\imath(x_1 - x_2)\wedge(x_2 - x_3). \tag{2.30c}$$

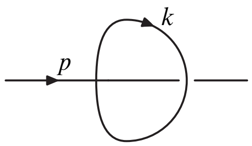
The interaction is real and positive[1]:

$$\begin{aligned}\int\prod_{i=1}^{4} dx_i\,\phi(x_i)\,\delta(x_1 - x_2 + x_3 - x_4)e^{\imath\varphi}\\ = \int dk\left(\int dxdy\,\phi(x)\phi(y)e^{\imath k(x-y)+\imath x\wedge y}\right)^2 \in \mathbb{R}_+.\end{aligned} \tag{2.31}$$

It is also translation invariant as equation (2.30c) shows.

The property 2.1 implies that the propagator is the usual one: $\hat{C}(p) = 1/(p^2+m^2)$.

2.1.3 UV/IR mixing. The non-locality of the $\star$-product allows to understand the discovery of Minwalla, Van Raamsdonk and Seiberg [26]. They showed that not only the model (2.28) is not finite in the UV but also it exhibits a new type of divergences making it non-renormalisable. In the article [8], Filk computed the Feynman rules corresponding to (2.28). He showed that the planar amplitudes equal the commutative ones whereas the non-planar ones give rise to oscillations coupling the internal and external legs. A typical example is the non-planar tadpole:



$$= \frac{\lambda}{12}\int\frac{d^4k}{(2\pi)^4}\frac{e^{ip_\mu k_\nu\Theta^{\mu\nu}}}{k^2+m^2}$$

$$= \frac{\lambda}{48\pi^2}\sqrt{\frac{m^2}{(\Theta p)^2}}K_1(\sqrt{m^2(\Theta p)^2}) \underset{p\to 0}{\simeq} p^{-2}. \tag{2.32}$$

If $p \neq 0$, this amplitude is finite but, for small p, it diverges like p^{-2}. In other words, if we put an ultraviolet cut-off Λ to the k-integral, the two limits $\Lambda \to \infty$ and $p \to 0$ do not commute. This is the UV/IR mixing phenomenon. A chain of non-planar tadpoles, inserted in bigger graphs, makes divergent any function (with six points or more for example). But this divergence is not local and cannot be absorbed in a mass redefinition. This is what makes the model non-renormalisable. We will see in Sections 3.4 and 4 that the UV/IR mixing results in a coupling of the different

[1]Another way to prove it is from (2.10), $\overline{\phi^{\star 4}} = \phi^{\star 4}$.

scales of the theory. We will also note that we should distinguish different types of mixing.

The UV/IR mixing was studied by several groups. First, Chepelev and Roiban [9] gave a power counting for different scalar models. They were able to identify the divergent graphs and to classify the divergences of the theories thanks to the topological data of the graphs. Then V. Gayral [27] showed that UV/IR mixing is present on all isospectral deformations (they consist in curved generalisations of the Moyal space and of the non-commutative torus). For this, he considered a scalar model (2.28) and discovered contributions to the effective action which diverge when the external momenta vanish. The UV/IR mixing is then a general characteristic of the non-commutative theories, at least on the deformations.

2.2 The Grosse–Wulkenhaar breakthrough

The situation remained so until H. Grosse and R. Wulkenhaar discovered a way to define a renormalisable non-commutative model. We will detail their result in Section 3 but the main message is the following. By adding an harmonic term to the Lagrangian (2.28),

$$\begin{aligned} S[\phi] = \int d^4x\Big(-\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) \\ + \frac{1}{2}m^2\,\phi \star \phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\Big)(x), \end{aligned} \tag{2.33}$$

where $\tilde{x} = 2\Theta^{-1}x$ and the metric is Euclidean, the model, in four dimensions, is renormalisable at all orders of perturbation [5]. We will see in Section 4 that this additional term give rise to an infrared cut-off and allows to decouple the different scales of the theory. The new model (2.33), we call it Φ^4_4, does not exhibit any mixing. This result is very important because it opens the way towards other non-commutative field theories. In the following, we will call *vulcanization*[2] the procedure consisting in adding a new term to a Lagrangian of a non-commutative theory in order to make it renormalisable.

The propagator C of this Φ^4-theory is the kernel of the inverse operator $-\Delta + \Omega^2\tilde{x}^2 + m^2$. It is known as the Mehler kernel ([28], [20])

$$C(x,y) = \frac{\Omega^2}{\theta^2\pi^2}\int_0^\infty \frac{dt}{\sinh^2(2\widetilde{\Omega}t)}\, e^{-\frac{\widetilde{\Omega}}{2}\coth(2\widetilde{\Omega}t)(x-y)^2 - \frac{\widetilde{\Omega}}{2}\tanh(2\widetilde{\Omega}t)(x+y)^2 - m^2 t}. \tag{2.34}$$

Langmann and Szabo remarked that the quartic interaction with Moyal product is invariant under a duality transformation. It is a symmetry between momentum and

[2] Vulcanization: process aimed at hardening rubber or rubber material by treating it with sulfur at a high temperature.

direct space. The interaction part of the model (2.33) is (see equation (2.17))

$$S_{\text{int}}[\phi] = \int d^4x\, \frac{\lambda}{4}(\phi \star \phi \star \phi \star \phi)(x) \tag{2.35}$$

$$= \int \prod_{a=1}^{4} d^4x_a\, \phi(x_a)\, V(x_1, x_2, x_3, x_4) \tag{2.36}$$

$$= \int \prod_{a=1}^{4} \frac{d^4p_a}{(2\pi)^4}\, \hat{\phi}(p_a)\, \hat{V}(p_1, p_2, p_3, p_4) \tag{2.37}$$

with

$$\begin{aligned} &V(x_1, x_2, x_3, x_4) \\ &\quad = \frac{\lambda}{4} \frac{1}{\pi^4 \det \Theta} \delta(x_1 - x_2 + x_3 - x_4) \cos(2(\Theta^{-1})_{\mu\nu}(x_1^\mu x_2^\nu + x_3^\mu x_4^\nu)), \\ &\hat{V}(p_1, p_2, p_3, p_4) \\ &\quad = \frac{\lambda}{4}(2\pi)^4 \delta(p_1 - p_2 + p_3 - p_4) \cos\Big(\frac{1}{2}\Theta^{\mu\nu}(p_{1,\mu}p_{2,\nu} + p_{3,\mu}p_{4,\nu})\Big), \end{aligned}$$

where we used a *cyclic* Fourier transform: $\hat{\phi}(p_a) = \int dx\, e^{(-1)^a \imath p_a x_a} \phi(x_a)$. The transformation

$$\hat{\phi}(p) \leftrightarrow \pi^2 \sqrt{|\det \Theta|}\, \phi(x), \qquad p_\mu \leftrightarrow \tilde{x}_\mu \tag{2.38}$$

exchanges (2.36) and (2.37). In addition, the free part of the model (2.28) is not covariant under this duality. The vulcanization adds a term to the Lagrangian which restores the symmetry. The theory (2.33) is then covariant under the Langmann–Szabo duality:

$$S[\phi; m, \lambda, \Omega] \mapsto \Omega^2\, S\Big[\phi; \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}\Big]. \tag{2.39}$$

By symmetry, the parameter Ω is confined in $[0, 1]$. Let us note that for $\Omega = 1$, the model is invariant.

The interpretation of that harmonic term is not yet clear. But the vulcanization procedure already allowed to prove the renormalisability of several other models on Moyal spaces such that ϕ^4_2 [29], $\phi^3_{2,4}$ [30], [31] and the LSZ models [11], [32], [33]. These last are of the type

$$S[\phi] = \int d^nx \Big(\frac{1}{2}\bar{\phi} \star (-\partial_\mu + \tilde{x}_\mu + m)^2 \phi + \frac{\lambda}{4}\bar{\phi} \star \phi \star \bar{\phi} \star \phi\Big)(x). \tag{2.40}$$

By comparison with (2.33), one notes that here the additional term is formally equivalent to a fixed magnetic background. Deep is the temptation to interpret it as such. This model is invariant under the above duality and is exactly soluble. Let us remark that the complex interaction in (2.40) makes the Langmann–Szabo duality more natural. It does not need a cyclic Fourier transform. The ϕ^3 have been studied at $\Omega = 1$ where they also exhibit a soluble structure.

2.3 The non-commutative Gross–Neveu model

Apart from the Φ^4_4, the modified Bosonic LSZ model [3] and supersymmetric theories, we now know several renormalisable non-commutative field theories. Nevertheless they either are super-renormalisable (Φ^4_2 [29]) or (and) studied at a special point in the parameter space where they are solvable (Φ^3_2, Φ^3_4 [30], [31], the LSZ models [11], [32], [33]). Although only logarithmically divergent for parity reasons, the non-commutative Gross–Neveu model is a just renormalisable quantum field theory as Φ^4_4. One of its main interesting features is that it can be interpreted as a non-local Fermionic field theory in a constant magnetic background. Then apart from strengthening the "vulcanization" procedure to get renormalisable non-commutative field theories, the Gross–Neveu model may also be useful for the study of the quantum Hall effect. It is also a good first candidate for a constructive study [34] of a non-commutative field theory as Fermionic models are usually easier to construct. Moreover its commutative counterpart being asymptotically free and exhibiting dynamical mass generation [35], [36], [37], a study of the physics of this model would be interesting.

The non-commutative Gross–Neveu model (GN^2_Θ) is a Fermionic quartically interacting quantum field theory on the Moyal plane $\mathbb{R}^2_\theta$. The skew-symmetric matrix Θ is

$$\Theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}. \tag{2.41}$$

The action is

$$S[\bar\psi,\psi] = \int dx \left(\bar\psi\left(-\imath\partial\!\!\!/ + \Omega\tilde{x}\!\!\!/ + m + \mu\,\gamma_5\right)\psi + V_{\mathrm{o}}(\bar\psi,\psi) + V_{\mathrm{no}}(\bar\psi,\psi)\right)(x), \tag{2.42}$$

where $\tilde{x} = 2\Theta^{-1}x$, $\gamma_5 = \imath\gamma^0\gamma^1$ and $V = V_{\mathrm{o}} + V_{\mathrm{no}}$ is the interaction part given hereafter. The μ-term appears at two-loop order. We use a Euclidean metric and the Feynman convention $a\!\!\!/ = \gamma^\mu a_\mu$. The γ^0 and γ^1 matrices form a two-dimensional representation of the Clifford algebra $\{\gamma^\mu,\gamma^\nu\} = -2\delta^{\mu\nu}$. Let us remark that the γ^μ's are then skew-Hermitian: $\gamma^{\mu\dagger} = -\gamma^\mu$.

Propagator. The propagator corresponding to the action (2.42) is given by the following lemma.

Lemma 2.4 (Propagator [20]). *The propagator of the Gross–Neveu model is*

$$\begin{aligned} C(x,y) &= \int d\mu_C(\bar\psi,\psi)\,\psi(x)\bar\psi(y) = \left(-\imath\partial\!\!\!/ + \Omega\tilde{x}\!\!\!/ + m\right)^{-1}(x,y) \qquad (2.43)\\ &= \int_0^\infty dt\, C(t;x,y), \end{aligned}$$

$$C(t;x,y) = -\frac{\Omega}{\theta\pi}\frac{e^{-tm^2}}{\sinh(2\widetilde{\Omega}t)}\, e^{-\frac{\widetilde{\Omega}}{2}\coth(2\widetilde{\Omega}t)(x-y)^2+\imath\Omega x\wedge y} \\ \times\left\{\imath\widetilde{\Omega}\coth(2\widetilde{\Omega}t)(\not{x}-\not{y})+\Omega(\tilde{\not{x}}-\tilde{\not{y}})-m\right\}e^{-2\imath\Omega t\gamma\Theta^{-1}\gamma} \tag{2.44}$$

with $\widetilde{\Omega}=\frac{2\Omega}{\theta}$ *et* $x\wedge y=2x\Theta^{-1}y$.

We also have $e^{-2\imath\Omega t\gamma\Theta^{-1}\gamma}=\cosh(2\widetilde{\Omega}t)\mathbb{1}_2-\imath\frac{\theta}{2}\sinh(2\widetilde{\Omega}t)\gamma\Theta^{-1}\gamma$.

If we want to study a N-*color* model, we can consider a propagator diagonal in these color indices.

Interactions. Concerning the interaction part V, recall that (see Corollary 2.3) for any f_1, f_2, f_3, f_4 in $\mathcal{A}_\Theta$,

$$\int dx\ (f_1\star f_2\star f_3\star f_4)(x) \\ = \frac{1}{\pi^2\det\Theta}\int\prod_{j=1}^{4}dx_j\, f_j(x_j)\,\delta(x_1-x_2+x_3-x_4)e^{-\imath\varphi}, \tag{2.45}$$

$$\varphi=\sum_{i<j=1}^{4}(-1)^{i+j+1}x_i\wedge x_j. \tag{2.46}$$

This product is non-local and only invariant under cyclic permutations of the fields. Then, contrary to the commutative Gross–Neveu model, for which there exits only one spinorial interaction, the GN^2_Θ model has, at least, six different interactions: the *orientable* ones

$$V_{\mathrm{o}}=\frac{\lambda_1}{4}\int dx\ \left(\bar\psi\star\psi\star\bar\psi\star\psi\right)(x) \tag{2.47a}$$

$$+\frac{\lambda_2}{4}\int dx\ \left(\bar\psi\star\gamma^\mu\psi\star\bar\psi\star\gamma_\mu\psi\right)(x) \tag{2.47b}$$

$$+\frac{\lambda_3}{4}\int dx\ \left(\bar\psi\star\gamma_5\psi\star\bar\psi\star\gamma_5\psi\right)(x), \tag{2.47c}$$

where ψ's and $\bar\psi$'s alternate, and the *non-orientable* ones

$$V_{\mathrm{no}}=\frac{\lambda_4}{4}\int dx\ \left(\psi\star\bar\psi\star\bar\psi\star\psi\right)(x) \tag{2.48a}$$

$$+\frac{\lambda_5}{4}\int dx\ \left(\psi\star\gamma^\mu\bar\psi\star\bar\psi\star\gamma_\mu\psi\right)(x) \tag{2.48b}$$

$$+\frac{\lambda_6}{4}\int dx\ \left(\psi\star\gamma_5\bar\psi\star\bar\psi\star\gamma_5\psi\right)(x). \tag{2.48c}$$

All these interactions have the same x kernel thanks to the equation (2.45). The reason for which we call these interactions orientable or not will become clear in Section 4.

3 Multi-scale analysis in the matrix basis

The matrix basis is a basis for Schwartz-class functions. In this basis, the Moyal product becomes a simple matrix product. Each field is then represented by an infinite matrix [25], [29], [38].

3.1 A dynamical matrix model

3.1.1 From the direct space to the matrix basis. In the matrix basis, the action (2.33) takes the form

$$S[\phi] = (2\pi)^{D/2}\sqrt{\det\Theta}\Big(\frac{1}{2}\phi\Delta\phi + \frac{\lambda}{4}\operatorname{Tr}\phi^4\Big), \tag{3.1}$$

where $\phi = \phi_{mn}$, $m, n \in \mathbb{N}^{D/2}$ and

$$\begin{aligned}\Delta_{mn,kl} = \sum_{i=1}^{D/2}\Big(\mu_0^2 + \frac{2}{\theta}(m_i + n_i + 1)\Big)\delta_{ml}\delta_{nk} \\ - \frac{2}{\theta}(1-\Omega^2)\Big(\sqrt{(m_i+1)(n_i+1)}\,\delta_{m_i+1,l_i}\delta_{n_i+1,k_i} \\ + \sqrt{m_i n_i}\,\delta_{m_i-1,l_i}\delta_{n_i-1,k_i}\Big)\prod_{j\neq i}\delta_{m_j l_j}\delta_{n_j k_j}.\end{aligned} \tag{3.2}$$

The (four-dimensional) matrix Δ represents the quadratic part of the Lagrangian. The first difficulty to study the matrix model (3.1) is the computation of its propagator G defined as the inverse of Δ:

$$\sum_{r,s\in\mathbb{N}^{D/2}} \Delta_{mn;rs}G_{sr;kl} = \sum_{r,s\in\mathbb{N}^{D/2}} G_{mn;rs}\Delta_{sr;kl} = \delta_{ml}\delta_{nk}. \tag{3.3}$$

Fortunately, the action is invariant under $\mathrm{SO}(2)^{D/2}$ thanks to the form (2.1) of the Θ matrix. It implies a conservation law

$$\Delta_{mn,kl} = 0 \iff m + k \neq n + l. \tag{3.4}$$

The result is ([5], [29])

$$\begin{aligned}G_{m,m+h;l+h,l} &= \frac{\theta}{8\Omega}\int_0^1 d\alpha\,\frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega}+(\frac{D}{4}-1)}}{(1+C\alpha)^{\frac{D}{2}}}\prod_{s=1}^{\frac{D}{2}} G^{(\alpha)}_{m^s,m^s+h^s;l^s+h^s,l^s},\\ G^{(\alpha)}_{m,m+h;l+h,l} \\ &= \left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{m+l+h}\sum_{u=\max(0,-h)}^{\min(m,l)}\mathcal{A}(m,l,h,u)\left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\right)^{m+l-2u},\end{aligned} \tag{3.5}$$

where $\mathcal{A}(m,l,h,u) = \sqrt{\binom{m}{m-u}\binom{m+h}{m-u}\binom{l}{l-u}\binom{l+h}{l-u}}$ and C is a function in Ω: $C(\Omega) = \frac{(1-\Omega)^2}{4\Omega}$. The main advantage of the matrix basis is that it simplifies the interaction part: $\phi^{\star 4}$ becomes $\operatorname{Tr}\phi^4$. But the propagator becomes very compllicated.

Let us remark that the matrix model (3.1) is *dynamical*: its quadratic part is not trivial. Usually, matrix models are *local*.

Definition 3.1. A matrix model is called *local* if $G_{mn;kl} = G(m,n)\delta_{ml}\delta_{nk}$ and *non-local* otherwise.

In the matrix theories, the Feynman graphs are ribbon graphs. The propagator $G_{mn;kl}$ is then represented by the Figure 1. In a local matrix model, the propagator

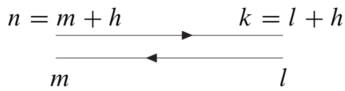
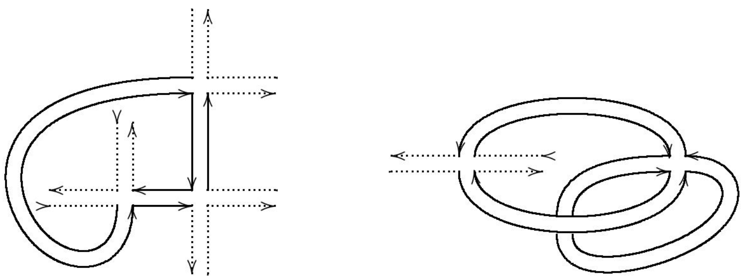


Figure 1. Matrix propagator.

preserves the index values along the trajectories (simple lines).

3.1.2 Topology of ribbon graphs. The power counting of a matrix model depends on the topological data of its graphs. The figure 2 gives two examples of ribbon graphs. Each ribbon graph may be drawn on a two-dimensional manifold. Actually



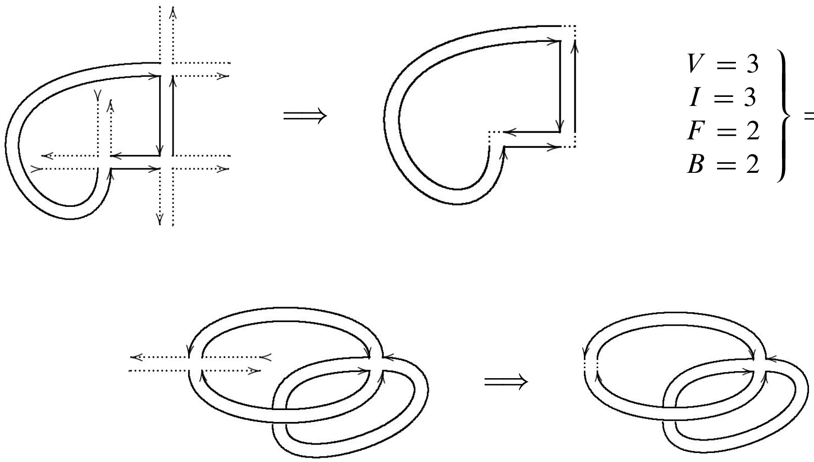
(a) Planar. (b) Non-planar.

Figure 2. Ribbon graphs.

each graph defines the surface on which it is drawn. Let a graph G with V vertices, I internal propagators (double lines) and F faces (made of simple lines). The Euler characteristic

$$\chi = 2 - 2g = V - I + F \tag{3.6}$$

gives the genus g of the manifold. One can make this clear by passing to the *dual graph*. The dual of a given graph G is obtained by exchanging faces and vertices. The dual graphs of the Φ^4-theory are tesselations of the surfaces on which they are drawn. Moreover each direct face broken by external legs becomes, in the dual graph, a *puncture*. If among the F faces of a graph, B are broken, this graph may be drawn on a surface of genus $g = 1 - \frac{1}{2}(V - I + F)$ with B punctures. The figure 3 gives the topological data of the graphs of the figure 2.



$$\left.\begin{array}{l} V = 3 \\ I = 3 \\ F = 2 \\ B = 2 \end{array}\right\} \Longrightarrow g = 0$$

$$\left.\begin{array}{l} V = 2 \\ I = 3 \\ F = 1 \\ B = 1 \end{array}\right\} \Longrightarrow g = 1$$

Figure 3. Topological data of ribbon graphs.

3.2 Multi-scale analysis

In [7] V. R., F. V.-T. and R. Wulkenhaar used the multi-scale analysis to reprove the power counting of the non-commutative Φ^4-theory.

3.2.1 Bounds on the propagator. Let G a ribbon graph of the Φ^4_4-theory with N external legs, V vertices, I internal lines and F faces. Its genus is then $g = 1 - \frac{1}{2}(V - I + F)$. Four indices $\{m, n; k, l\} \in \mathbb{N}^2$ are associated to each internal line of the graph and two indices to each external line, that is to say $4I + 2N = 8V$ indices. But, at each vertex, the left index of a ribbon equals the right one of the neighbour ribbon. This gives rise to $4V$ independent identifications which allows to write each index in terms of a set $\mathcal{I}$ made of $4V$ indices, four per vertex, for example the left index of each half-ribbon.

The graph amplitude is then

$$A_G = \sum_{\mathcal{I}} \prod_{\delta \in G} G_{m_\delta(\mathcal{I}),n_\delta(\mathcal{I});k_\delta(\mathcal{I}),l_\delta(\mathcal{I})}\ \delta_{m_\delta - l_\delta, n_\delta - k_\delta}, \tag{3.7}$$

where the four indices of the propagator G of the line δ are functions of $\mathcal{I}$ and written $\{m_\delta(\mathcal{I}), n_\delta(\mathcal{I}); k_\delta(\mathcal{I}), l_\delta(\mathcal{I})\}$. We decompose each propagator, given by (3.5):

$$G = \sum_{i=0}^{\infty} G^i \quad \text{thanks to} \quad \int_0^1 d\alpha = \sum_{i=1}^{\infty} \int_{M^{-2i}}^{M^{-2(i-1)}} d\alpha,\ M > 1. \tag{3.8}$$

We have an associated decomposition for each amplitude:

$$A_G = \sum_{\mu} A_{G,\mu}, \tag{3.9}$$

$$A_{G,\mu} = \sum_{\mathcal{I}} \prod_{\delta \in G} G^{i_\delta}_{m_\delta(\mathcal{I}),n_\delta(\mathcal{I});k_\delta(\mathcal{I}),l_\delta(\mathcal{I})}\ \delta_{m_\delta(\mathcal{I}) - l_\delta(\mathcal{I}), n_\delta(\mathcal{I}) - k_\delta(\mathcal{I})}, \tag{3.10}$$

where $\mu = \{i_\delta\}$ runs over the all possible assignments of a positive integer i_δ to each line δ. We proved the following four propositions.

Proposition 3.1. *For M large enough, there exists a constant K such that, for $\Omega \in [0.5, 1]$, we have the uniform bound*

$$G^i_{m,m+h;l+h,l} \leqslant KM^{-2i} e^{-\frac{\Omega}{3} M^{-2i} \|m+l+h\|}. \tag{3.11}$$

Proposition 3.2. *For M large enough, there exist two constants K and K_1 such that, for $\Omega \in [0.5, 1]$, we have the uniform bound*

$$\begin{aligned} &G^i_{m,m+h;l+h,l} \\ &\quad \leqslant KM^{-2i} e^{-\frac{\Omega}{4} M^{-2i} \|m+l+h\|} \\ &\quad \prod_{s=1}^{\frac{D}{2}} \min\left(1, \left(\frac{K_1 \min(m^s, l^s, m^s + h^s, l^s + h^s)}{M^{2i}}\right)^{\frac{|m^s - l^s|}{2}}\right). \end{aligned} \tag{3.12}$$

This bound allows to prove that the only diverging graphs have either a constant index along the trajectories or a total jump of 2.

Proposition 3.3. *For M large enough, there exists a constant K such that, for $\Omega \in \left[\frac{2}{3}, 1\right]$, we have the uniform bound*

$$\sum_{l=-m}^{p} G^i_{m,p-l,p,m+l} \leqslant KM^{-2i}\ e^{-\frac{\Omega}{4} M^{-2i} (\|p\| + \|m\|)}. \tag{3.13}$$

This bound shows that the propagator is almost local in the following sense: with m fixed, the sum over l does not cost anything (see Figure 1). Nevertheless the sums we will have to perform are entangled (a given index may enter different propagators) so that we need the following proposition.

Proposition 3.4. *For M large enough, there exists a constant K such that, for $\Omega \in \left[\frac{2}{3}, 1\right]$, we have the uniform bound*

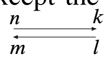
$$\sum_{l=-m}^{\infty} \max_{p \geqslant \max(l,0)} G^{i}_{m,p-l;p,m+l} \leqslant K M^{-2i} e^{-\frac{\Omega}{36} M^{-2i} \|m\|} . \tag{3.14}$$

We refer to [7] for the proofs of these four propositions.

3.2.2 Power counting. About half of the $4V$ indices initially associated to a graph is determined by the external indices and the δ-functions in (3.7). The other indices are summation indices. The power counting consists in finding which sums cost M^{2i} and which cost $\mathcal{O}(1)$ thanks to (3.13). The M^{2i} factor comes from (3.11) after a summation over an index[3] $m \in \mathbb{N}^2$,

$$\sum_{m^1,m^2=0}^{\infty} e^{-cM^{-2i}(m^1+m^2)} = \frac{1}{(1-e^{-cM^{-2i}})^2} = \frac{M^{4i}}{c^2}(1+\mathcal{O}(M^{-2i})). \tag{3.15}$$

We first use the δ-functions as much as possible to reduce the set $\mathcal{I}$ to a true minimal set $\mathcal{I}'$ of independent indices. For this, it is convenient to use the dual graphs where the resolution of the δ-functions is equivalent to a usual momentum routing.

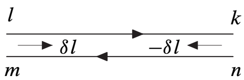
The dual graph is made of the same propagators as the direct graph except the position of their indices. Whereas in the original graph we have $G_{mn;kl} =$ 

the position of the indices in a dual propagator is

$$G_{mn;kl} = \quad$$ 

. (3.16)

The conservation $\delta_{l-m,-(n-k)}$ in (3.7) implies that the difference $l-m$ is conserved along the propagator. These differences behave like an *angular momentum* and the conservation of the differences $\ell = l - m$ and $-\ell = n - k$ is nothing else than the conservation of the angular momentum thanks to the symmetry SO(2) × SO(2) of the action (3.1):



$$l = m + \ell, \quad n = k + (-\ell). \tag{3.17}$$

The cyclicity of the vertices implies the vanishing of the sum of the angular momenta entering a vertex. Thus the angular momentum in the dual graph behaves exactly like the usual momentum in ordinary Feynman graphs.

We know that the number of independent momenta is exactly the number L' ($= I - V' + 1$ for a connected graph) of loops in the dual graph. Each index at a (dual) vertex is then given by a unique *reference index* and a sum of momenta. If the

[3] Recall that each index is in fact made of two indices, one for each symplectic pair of $\mathbb{R}^4_\theta$.

dual vertex under consideration is an external one, we choose an external index for the reference index. The reference indices in the dual graph correspond to the loop indices in the direct graph. The number of summation indices is then $V'-B+L' = I+(1-B)$ where $B \geqslant 0$ is the number of broken faces of the direct graph or the number of external vertices in the dual graph.

By using a well-chosen order on the lines, an optimized tree and a L^1-L^∞ bound, one can prove that the summation over the angular momenta does not cost anything thanks to (3.13). Recall that a connected component is a subgraph for which all internal lines have indices greater than all its external ones. The power counting is then:

$$A_G \leqslant K'^V \sum_\mu \prod_{i,k} M^{\omega(G^i_k)} \tag{3.18}$$

with

$$\begin{aligned} \omega(G^i_k) &= 4(V'_{i,k} - B_{i,k}) - 2I_{i,k} = 4(F_{i,k} - B_{i,k}) - 2I_{i,k} \\ &= (4 - N_{i,k}) - 4(2g_{i,k} + B_{i,k} - 1) \end{aligned} \tag{3.19}$$

and $N_{i,k}$, $V_{i,k}$, $I_{i,k} = 2V_{i,k} - \frac{N_{i,k}}{2}$, $F_{i,k}$ and $B_{i,k}$ are respectively the numbers of external legs, of vertices, of (internal) propagators, of faces and broken faces of the connected component G^i_k; $g_{i,k} = 1 - \frac{1}{2}(V_{i,k} - I_{i,k} + F_{i,k})$ is its genus. We have

Theorem 3.5. *The sum over the scales attributions μ converges if $\omega(G^i_k) < 0$ for all i, k.*

We recover the power counting obtained in [4].

From this point on, renormalisability of ϕ^4_4 can proceed (however remark that it remains limited to $\Omega \in [0.5, 1]$ by the technical estimates such as (3.11); this limitation is overcome in the direct space method below).

The multiscale analysis allows to define the so-called effective expansion, in between the bare and the renormalised expansion, which is optimal, both for physical and for constructive purposes [34]. In this effective expansion only the subcontributions with all *internal* scales higher than all *external* scales have to be renormalised by counterterms of the form of the initial Lagrangian.

In fact only planar such subcontributions with a single external face must be renormalised by such counterterms. This follows simply from the Grosse–Wulkenhaar moves defined in [4]. These moves translate the external legs along the outer border of the planar graph, up to irrelevant corrections, until they all merge together into a term of the proper Moyal form, which is then absorbed in the effective constants definition. This requires only the estimates (3.11)–(3.14), which were checked numerically in [4].

In this way the relevant and marginal counterterms can be shown to be of the Moyal type, namely renormalise the parameters λ, m and Ω[4].

[4]The wave function renormalisation i.e. renormalisation of the $\partial_\mu\phi \star \partial^\mu\phi$ term can be absorbed in a rescaling of the field, called "field strength renromalization."

Notice that in the multiscale analysis there is no need for the relatively complicated use of Polchinski's equation [39] made in [4]. Polchinski's method, although undoubtedly very elegant for proving perturbative renormalisability does not seem directly suited to constructive purposes, even in the case of simple Fermionic models such as the commutative Gross Neveu model, see e.g. [40].

The BPHZ theorem itself for the renormalised expansion follows from finiteness of the effective expansion by developing the counterterms still hidden in the effective couplings. Its own finiteness can be checked e.g. through the standard classification of forests [34]. Let us however recall once again that in our opinion the effective expansion, not the renormalised one is the more fundamental object, both to describe the physics and to attack deeper mathematical problems, such as those of constructive theory [34], [41].

The matrix base simplifies very much at $\Omega = 1$, where the matrix propagator becomes diagonal, i.e. conserves exactly indices. This property has been used for the general proof that the beta function of the theory vanishes in the ultraviolet regime [24], leading to the exciting perspective of a full non-perturbative construction of the model.

3.3 Propagators on non-commutative space

We give here the results we get in [20]. In this article, we computed the x-space and matrix basis kernels of operators which generalize the Mehler kernel (2.34). Then we proceeded to a study of the scaling behaviours of these kernels in the matrix basis. This work is useful to study the non-commutative Gross–Neveu model in the matrix basis.

3.3.1 Bosonic kernel. The following lemma generalizes the Mehler kernel [28].

Lemma 3.6. *Let H the operator*

$$H = \frac{1}{2}\big(- \Delta + \Omega^2 x^2 - 2\imath B(x_0 \partial_1 - x_1 \partial_0)\big). \tag{3.20}$$

The x-space kernel of e^{-tH} is

$$e^{-tH}(x, x') = \frac{\Omega}{2\pi \sinh \Omega t} e^{-A}, \tag{3.21}$$

$$A = \frac{\Omega \cosh \Omega t}{2 \sinh \Omega t}(x^2 + x'^2) - \frac{\Omega \cosh Bt}{\sinh \Omega t} x \cdot x' - \imath \frac{\Omega \sinh Bt}{\sinh \Omega t} x \wedge x'. \tag{3.22}$$

Remark. The Mehler kernel corresponds to $B = 0$. The limit $\Omega = B \to 0$ gives the usual heat kernel.

Lemma 3.7. *Let H be given by (3.20) with $\Omega(B) \to 2\Omega/\theta(2B\theta)$. Its inverse in the matrix basis is*

$$H^{-1}_{m,m+h;l+h,l} \tag{3.23}$$
$$= \frac{\theta}{8\Omega}\int_0^1 d\alpha\, \frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega}+(\frac{D}{4}-1)}}{(1+C\alpha)^{\frac{D}{2}}}(1-\alpha)^{-\frac{4B}{8\Omega}h}\prod_{s=1}^{\frac{D}{2}} G^{(\alpha)}_{m^s,m^s+h^s;l^s+h^s,l^s},$$
$$G^{(\alpha)}_{m,m+h;l+h,l}$$
$$= \left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{m+l+h}\sum_{u=\max(0,-h)}^{\min(m,l)} \mathcal{A}(m,l,h,u)\left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\right)^{m+l-2u},$$

where $\mathcal{A}(m,l,h,u) = \sqrt{\binom{m}{m-u}\binom{m+h}{m-u}\binom{l}{l-u}\binom{l+h}{l-u}}$ *and C is a function of Ω:* $C(\Omega) = \frac{(1-\Omega)^2}{4\Omega}$.

3.3.2 Fermionic kernel.

On the Moyal space, we modified the commutative Gross–Neveu model by adding a $\tilde{\not{x}}$ term (see Lemma 2.4). We have

$$G(x,y) = -\frac{\Omega}{\theta\pi}\int_0^\infty \frac{dt}{\sinh(2\tilde\Omega t)}\, e^{-\frac{\tilde\Omega}{2}\coth(2\tilde\Omega t)(x-y)^2+\imath\tilde\Omega x\wedge y} \tag{3.24}$$
$$\left\{\imath\tilde\Omega\coth(2\tilde\Omega t)(\not{x}-\not{y})+\Omega(\tilde{\not{x}}-\tilde{\not{y}})-\mu\right\}e^{-2\imath\tilde\Omega t\gamma^0\gamma^1}e^{-t\mu^2}.$$

It will be useful to express G in terms of commutators:

$$G(x,y) = -\frac{\Omega}{\theta\pi}\int_0^\infty dt\,\left\{\imath\tilde\Omega\coth(2\tilde\Omega t)\left[\not{x},\Gamma^t\right](x,y)\right.$$
$$\left.+\Omega\left[\tilde{\not{x}},\Gamma^t\right](x,y)-\mu\Gamma^t(x,y)\right\}e^{-2\imath\tilde\Omega t\gamma^0\gamma^1}e^{-t\mu^2}, \tag{3.25}$$

where

$$\Gamma^t(x,y) = \frac{1}{\sinh(2\tilde\Omega t)}\, e^{-\frac{\tilde\Omega}{2}\coth(2\tilde\Omega t)(x-y)^2+\imath\tilde\Omega x\wedge y} \tag{3.26}$$

with $\tilde\Omega = \frac{2\Omega}{\theta}$ and $x\wedge y = x^0y^1 - x^1y^0$.

We now give the expression of the Fermionic kernel (3.25) in the matrix basis. The inverse of the quadratic form

$$\Delta = p^2+\mu^2+\frac{4\Omega^2}{\theta^2}x^2+\frac{4B}{\theta}L_2 \tag{3.27}$$

is given by (3.23) above:

$$\Gamma_{m,m+h;l+h,l} = \frac{\theta}{8\Omega}\int_0^1 d\alpha\,\frac{(1-\alpha)^{\frac{\mu^2\theta}{8\Omega}-\frac12}}{(1+C\alpha)}\Gamma^\alpha_{m,m+h;l+h,l}, \tag{3.28}$$

$$\Gamma^{(\alpha)}_{m,m+h;l+h,l} = \left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{m+l+h} (1-\alpha)^{-\frac{Bh}{2\Omega}} \sum_{u=0}^{\min(m,l)} \mathcal{A}(m,l,h,u) \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}\,(1-\Omega)}\right)^{m+l-2u}. \tag{3.29}$$

The Fermionic propagator G (3.25) in the matrix basis may be deduced from the kernel (3.28). We just set $B=\Omega$, add the missing term with $\gamma^0\gamma^1$ and compute the action of $-\not{p}-\Omega\tilde{\not{x}}+\mu$ on Γ. We must then evaluate $[x^\nu,\Gamma]$ in the matrix basis:

$$\begin{aligned}\left[x^0,\Gamma\right]_{m,n;k,l} = 2\pi\theta\sqrt{\frac{\theta}{8}}\Big\{&\sqrt{m+1}\Gamma_{m+1,n;k,l} - \sqrt{l}\Gamma_{m,n;k,l-1} + \sqrt{m}\Gamma_{m-1,n;k,l}\\ &- \sqrt{l+1}\Gamma_{m,n;k,l+1} + \sqrt{n+1}\Gamma_{m,n+1;k,l} - \sqrt{k}\Gamma_{m,n;k-1,l}\\ &+\sqrt{n}\Gamma_{m,n-1;k,l} - \sqrt{k+1}\Gamma_{m,n;k+1,l}\Big\},\end{aligned} \tag{3.30}$$

$$\begin{aligned}\left[x^1,\Gamma\right]_{m,n;k,l} = 2\imath\pi\theta\sqrt{\frac{\theta}{8}}\Big\{&\sqrt{m+1}\Gamma_{m+1,n;k,l} - \sqrt{l}\Gamma_{m,n;k,l-1} - \sqrt{m}\Gamma_{m-1,n;k,l}\\ &+ \sqrt{l+1}\Gamma_{m,n;k,l+1} - \sqrt{n+1}\Gamma_{m,n+1;k,l} + \sqrt{k}\Gamma_{m,n;k-1,l}\\ &+\sqrt{n}\Gamma_{m,n-1;k,l} - \sqrt{k+1}\Gamma_{m,n;k+1,l}\Big\}.\end{aligned} \tag{3.31}$$

This allows to prove:

Lemma 3.8. *Let $G_{m,n;k,l}$ the kernel, in the matrix basis, of the operator $\left(\not{p}+\Omega\tilde{\not{x}}+\mu\right)^{-1}$. We have:*

$$G_{m,n;k,l} = -\frac{2\Omega}{\theta^2\pi^2}\int_0^1 d\alpha\, G^\alpha_{m,n;k,l}, \tag{3.32}$$

$$\begin{aligned}G^\alpha_{m,n;k,l} = &\left(\imath\tilde{\Omega}\frac{2-\alpha}{\alpha}\left[\not{x},\Gamma^\alpha\right]_{m,n;k,l} + \Omega\left[\tilde{\not{x}},\Gamma^\alpha\right]_{m,n;k,l} - \mu\,\Gamma^\alpha_{m,n;k,l}\right)\\ &\times\left(\frac{2-\alpha}{2\sqrt{1-\alpha}}\mathbb{1}_2 - \imath\frac{\alpha}{2\sqrt{1-\alpha}}\gamma^0\gamma^1\right),\end{aligned} \tag{3.33}$$

where Γ^α is given by (3.29) and the commutators by the formulas (3.30) and (3.31).

The first two terms in the equation (3.33) contain commutators and will be gathered under the name $G^{\alpha,\mathrm{comm}}_{m,n;k,l}$. The last term will be called $G^{\alpha,\mathrm{mass}}_{m,n;k,l}$:

$$\begin{aligned}G^{\alpha,\mathrm{comm}}_{m,n;k,l} = &\left(\imath\tilde{\Omega}\frac{2-\alpha}{\alpha}\left[\not{x},\Gamma^\alpha\right]_{m,n;k,l} + \Omega\left[\tilde{\not{x}},\Gamma^\alpha\right]_{m,n;k,l}\right)\\ &\times\left(\frac{2-\alpha}{2\sqrt{1-\alpha}}\mathbb{1}_2 - \imath\frac{\alpha}{2\sqrt{1-\alpha}}\gamma^0\gamma^1\right),\end{aligned} \tag{3.34}$$

$$G^{\alpha,\text{mass}}_{m,n;k,l} = -\mu\, \Gamma^{\alpha}_{m,n;k,l} \times \left(\frac{2-\alpha}{2\sqrt{1-\alpha}} \mathbb{1}_2 - \imath \frac{\alpha}{2\sqrt{1-\alpha}} \gamma^0 \gamma^1 \right). \tag{3.35}$$

3.3.3 Bounds. We use the multi-scale analysis to study the behaviour of the propagator (3.33) and revisit more finely the bounds (3.11) to (3.14). In a slice i, the propagator is

$$\Gamma^i_{m,m+h,l+h,l} = \frac{\theta}{8\Omega} \int_{M^{-2i}}^{M^{-2(i-1)}} d\alpha\, \frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega} - \frac{1}{2}}}{(1+C\alpha)} \Gamma^{(\alpha)}_{m,m+h;l+h,l}; \tag{3.36}$$

$$G_{m,n;k,l} = \sum_{i=1}^{\infty} G^i_{m,n;k,l}, \quad G^i_{m,n;k,l} = -\frac{2\Omega}{\theta^2\pi^2} \int_{M^{-2i}}^{M^{-2(i-1)}} d\alpha\, G^{\alpha}_{m,n;k,l}. \tag{3.37}$$

Let $h = n - m$ and $p = l - m$. Without loss of generality, we assume $h \geqslant 0$ and $p \geqslant 0$. Then the smallest index among m, n, k, l is m and the biggest is $k = m + h + p$. We have:

Theorem 3.9. *Under the assumptions $h = n - m \geqslant 0$ and $p = l - m \geqslant 0$, there exists $K, c \in \mathbb{R}_+$ (c depends on Ω) such that the propagator of the non-commutative Gross–Neveu model in a slice i obeys the bound*

$$\begin{aligned} |G^{i,\text{comm}}_{m,n;k,l}| \leqslant KM^{-i} \Bigg(&\chi(\alpha k > 1) \frac{\exp\{-\frac{cp^2}{1+kM^{-2i}} - \frac{cM^{-2i}}{1+k}(h - \frac{k}{1+C})^2\}}{(1 + \sqrt{kM^{-2i}})} \\ &+ \min(1, (\alpha k)^p) e^{-ckM^{-2i} - cp} \Bigg). \end{aligned} \tag{3.38}$$

The mass term is slightly different:

$$\begin{aligned} |G^{i,\text{mass}}_{m,n;k,l}| \leqslant KM^{-2i} \Bigg(&\chi(\alpha k > 1) \frac{\exp\{-\frac{cp^2}{1+kM^{-2i}} - \frac{cM^{-2i}}{1+k}(h - \frac{k}{1+C})^2\}}{1 + \sqrt{kM^{-2i}}} \\ &+ \min(1, (\alpha k)^p) e^{-ckM^{-2i} - cp} \Bigg). \end{aligned} \tag{3.39}$$

Remark. We can redo the same analysis for the Φ^4 propagator and get

$$G^i_{m,n;k,l} \leqslant KM^{-2i} \min(1, (\alpha k)^p)\, e^{-c(M^{-2i}k + p)} \tag{3.40}$$

which allows to recover the bounds (3.11) to (3.14).

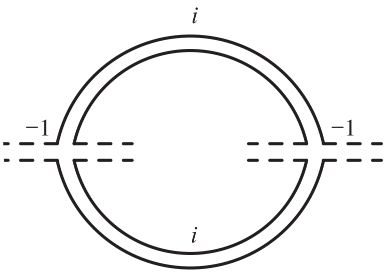
3.4 Propagators and renormalisability

Let us consider the propagator (3.32) of the non-commutative Gross–Neveu model. We saw in Section 3.3.3 that there exists two regions in the space of indices where the propagator behaves very differently. In one of them it behaves as the Φ^4 propagator and leads then to the same power counting. In the critical region, we have

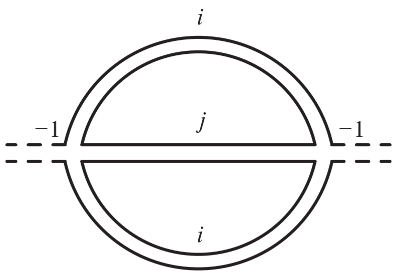
$$G^i \leqslant K \frac{M^{-i}}{1+\sqrt{kM^{-2i}}}\, e^{-\frac{cp^2}{1+kM^{-2i}}-\frac{cM^{-2i}}{1+k}(h-\frac{k}{1+C})^2}. \tag{3.41}$$

The point is that such a propagator does not allow to sum two reference indices with a unique line. This fact was useful in the proof of the power counting of the Φ^4 model. This leads to a *renormalisable* UV/IR mixing.

Let us consider the graph in Figure 4 (b) where two of the internal lines bear an index $i \gg 1$ and the third one an index $j < i$. The propagator (3.32) obeys the bound (3.13) in Proposition 3.3 which means that it is almost local. We only have to sum over one index per internal face.



(a) At scale i.



(b) At scale j.

Figure 4. Sunset graph.

In the graph of the figure 4 (a), if the two lines inside are true external ones, the graph has two broken faces and there is no index to sum over. Then by using Proposition 3.11 we get $A_G \leqslant M^{-2i}$. The sum over i converges and we have the same behaviour as the Φ^4-theory, that is to say the graphs with $B \geqslant 2$ broken faces are finite. But if these two lines belongs to a line of scale $j < i$ (see Figure 4 (b)), the result is different. Indeed, at scale i, we recover the graph of Figure 4 (a). To maintain the previous result (M^{-2i}), we should sum the two indices corresponding to the internal faces with the propagator of scale j. This is not possible. Instead we have

$$\sum_{k,h} M^{-2i-j}\, e^{-M^{-2i}k}\, \frac{e^{-\frac{cM^{-2j}}{1+k}(h-\frac{k}{1+C})^2}}{1+\sqrt{kM^{-2j}}} \leqslant KM^j. \tag{3.42}$$

The sum over i diverges logarithmically. The graph of Figure 4 (a) converges if it is linked to true external legs et diverges if it is a subgraph of a graph at a lower scale.

The power counting depends on the scales lower than the lowest scale of the graph. It cannot then be factorized into the connected components: this is UV/IR mixing.

Let us remark that the graph of Figure 4 (a) is not renormalisable by a counter-term in the Lagrangian. Its logarithmic divergence cannot be absorbed in a redefinition of a coupling constant. Fortunately the renormalisation of the two-point graph of Figure 4 (b) makes the four-point subdivergence finite [21]. This makes the non-commutative Gross–Neveu model renormalisable.

4 Direct space

We want now to explain how the power counting analysis can be performed in direct space, and the "Moyality" of the necessary counterterms can be checked by a Taylor expansion which is a generalization of the one used in direct commutative space.

In the commutative case there is translation invariance, hence each propagator depends on a single difference variable which is short in the ultraviolet regime; in the non-commutative case the propagator depends both of the difference of end positions, which is again short in the uv regime, but also of the sum which is long in the uv regime, considering the explicit form (2.34) of the Mehler kernel.

This distinction between short and long variables is at the basis of the power counting analysis in direct space.

4.1 Short and long variables

Let G be an arbitrary connected graph. The amplitude associated with this graph is in direct space (with hopefully self-explaining notations):

$$A_G = \int \prod_{v,i=1,\dots,4} dx_{v,i} \prod_l dt_l \tag{4.1}$$

$$\prod_v \Big[\delta(x_{v,1} - x_{v,2} + x_{v,3} - x_{v,4}) e^{\imath \sum_{i<j} (-1)^{i+j+1} x_{v,i}\theta^{-1} x_{v,j}}\Big] \prod_l C_l ,$$

$$C_l = \frac{\Omega^2}{[2\pi \sinh(\Omega t_l)]^2} e^{-\frac{\Omega}{2}\coth(\Omega t_l)(x_{v,i(l)}^2 + x_{v',i'(l)}^2) + \frac{\Omega}{\sinh(\Omega t_l)} x_{v,i(l)} \cdot x_{v',i'(l)} - \mu_0^2 t_l} .$$

For each line l of the graph joining positions $x_{v,i(l)}$ and $x_{v',i'(l)}$, we choose an orientation and we define the "short" variable $u_l = x_{v,i(l)} - x_{v',i'(l)}$ and the "long" variable $v_l = x_{v,i(l)} + x_{v',i'(l)}$.

With these notations, defining $\Omega t_l = \alpha_l$, the propagators in our graph can be written as

$$\int_0^\infty \prod_l \frac{\Omega d\alpha_l}{[2\pi \sinh(\alpha_l)]^2} e^{-\frac{\Omega}{4}\coth(\frac{\alpha_l}{2}) u_l^2 - \frac{\Omega}{4}\tanh(\frac{\alpha_l}{2}) v_l^2 - \frac{\mu_0^2}{\Omega}\alpha_l} . \tag{4.2}$$

As in matrix space we can slice each propagator according to the size of its α parameter and obtain the multiscale representation of each Feynman amplitude:

$$A_G = \sum_{\mu} A_{G,\mu}, \quad A_{G,\mu} = \int \prod_{v,i=1,\ldots,4} dx_{v,i} \prod_l C_l^{i_\mu(l)}(u_l, v_l)$$
$$\prod_v \Big[\delta(x_{v,1} - x_{v,2} + x_{v,3} - x_{v,4}) e^{\imath \sum_{i<j}(-1)^{i+j+1} x_{v,i}\theta^{-1}x_{v,j}}\Big] \tag{4.3}$$

$$C^i(u,v) = \int_{M^{-2i}}^{M^{-2(i-1)}} \frac{\Omega d\alpha}{[2\pi \sinh(\alpha)]^2} e^{-\frac{\Omega}{4}\coth(\frac{\alpha}{2})u^2 - \frac{\Omega}{4}\tanh(\frac{\alpha}{2})v^2 - \frac{\mu_0^2}{\Omega}\alpha}, \tag{4.4}$$

where μ runs over scales attributions $\{i_\mu(l)\}$ for each line l of the graph, and the sliced propagator C^i in slice $i \in \mathbb{N}$ obeys the following crude bound.

Lemma 4.1. *For some constants K (large) and c (small):*

$$C^i(u,v) \leqslant KM^{2i} e^{-c[M^i\|u\| + M^{-i}\|v\|]} \tag{4.5}$$

(*which a posteriori justifies the terminology of "long" and "'short" variables*).

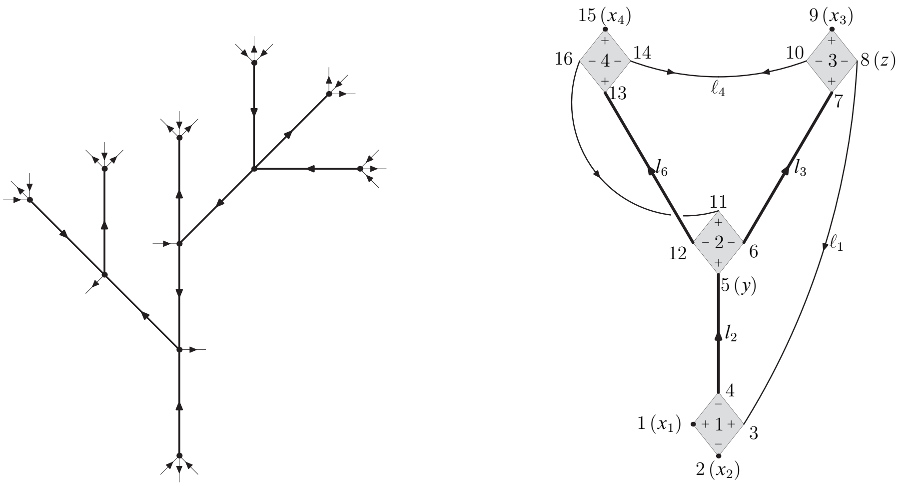
The proof is elementary.

4.2 Routing, Filk moves

4.2.1 Oriented graphs. We pick a tree T of lines of the graph, hence connecting all vertices, pick with a root vertex and build an *orientation* of all the lines of the graph in an inductive way. Starting from an arbitrary orientation of a field at the root of the tree, we climb in the tree and at each vertex of the tree we impose cyclic order to alternate entering and exiting tree lines and loop half-lines, as in Figure 5 (a). Then we look at the loop lines. If every loop lines consist in the contraction of an entering and an exiting line, the graph is called orientable. Otherwise we call it non-orientable as in Figure 5 (b).

4.2.2 Position routing. There are n δ-functions in an amplitude with n vertices, hence n linear equations for the $4n$ positions, one for each vertex. The *position routing* associated to the tree T solves this system by passing to another equivalent system of n linear equations, one for each branch of the tree. This is a triangular change of variables, of Jacobian 1. This equivalent system is obtained by summing the arguments of the δ-functions of the vertices in each branch. This change of variables is exactly the x-space analog of the resolution of momentum conservation called *momentum routing* in the standard physics literature of commutative field theory, except that one should now take care of the additional $\pm$ cyclic signs.

One can prove [3] that the rank of the system of δ-functions in an amplitude with n vertices is



(a) Orientation of a tree. (b) A non-orientable graph.

Figure 5. Orientation.

- $n-1$ if the graph is orientable,
- n if the graph is non-orientable.

The position routing change of variables is summarized by the following lemma:

Lemma 4.2 (Position routing). *We have, calling I_G the remaining integrand in (4.3),*

$$A_G = \int \Big[\prod_v \big[\delta(x_{v,1} - x_{v,2} + x_{v,3} - x_{v,4})\big]\Big]\, I_G(\{x_{v,i}\}) \tag{4.6}$$

$$= \int \prod_b \delta\Big(\sum_{l \in T_b \cup L_b} u_l + \sum_{l \in L_{b,+}} v_l - \sum_{l \in L_{b,-}} v_l + \sum_{f \in X_b} \varepsilon(f) x_f \Big) I_G(\{x_{v,i}\}),$$

where $\varepsilon(f)$ is ± 1 depending on whether the field f enters or exits the branch.

We can now use the system of δ-functions to eliminate variables. It is of course better to eliminate long variables as their integration costs a factor M^{4i} whereas the integration of a short variable brings M^{-4i}. Rough power counting, neglecting all oscillations of the vertices leads therefore, in the case of an orientable graph with N external fields, n internal vertices and $l = 2n - N/2$ internal lines at scale i to:

- a factor $M^{2i(2n-N/2)}$ coming from the M^{2i} factors for each line of scale i in (4.5),
- a factor $M^{-4i(2n-N/2)}$ for the $l = 2n - N/2$ short variables integrations,

- a factor $M^{4i(n-N/2+1)}$ for the long variables after eliminating $n-1$ of them using the δ-functions.

The total factor is therefore $M^{-(N-4)i}$, the ordinary scaling of ϕ_4^4, which means that only two and four point subgraphs ($N \leqslant 4$) diverge when i has to be summed.

In the non-orientable case, we can eliminate one additional long variable since the rank of the system of δ-functions is larger by one unit! Therefore we get a power counting bound M^{-Ni}, which proves that only *orientable* graphs may diverge.

In fact we of course know that not all *orientable* two and four point subgraphs diverge but only the planar ones with a single external face. (It is easy to check that all such planar graphs are indeed orientable.)

Since only these planar subgraphs with a single external face can be renormalised by Moyal counterterms, we need to prove that orientable, non-planar graphs or orientable planar graphs with several external faces have in fact a better power counting than this crude estimate. This can be done only by exploiting their vertices oscillations. We explain now how to do this with minimal effort.

4.2.3 Filk moves and rosettes. Following Filk [8], we can contract all lines of a spanning tree T and reduce G to a single vertex with "tadpole loops" called a "rosette graph". This rosette is a cycle (which is the border of the former tree) bearing loops lines on it (see Figure 6): Remark that the rosette can also be considered as a big

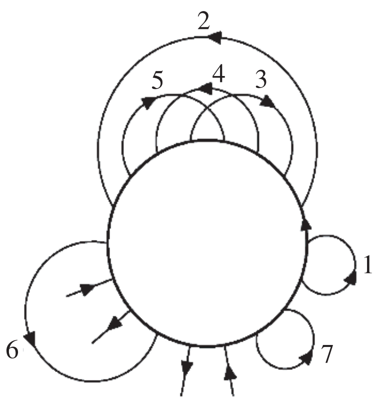


Figure 6. A rosette.

vertex, with $r = 2n+2$ fields, on which N are external fields with external variables x and $2n+2-N$ are loop fields for the corresponding $n+1-N/2$ loops. When the graph is orientable, the rosette is also orientable, which means that turning around the rosette the lines alternatively enter and exit. These lines correspond to the contraction of the fields on the border of the tree T before the Filk contraction, also called the "first Filk move".

4.2.4 Rosette factor. We start from the root and turn around the tree in the trigonometrical sense. We number separately all the fields as $1, \dots, 2n+2$ and all the tree lines as $1, \dots, n-1$ in the order they are met.

Lemma 4.3. *The rosette contribution after a complete first Filk reduction is exactly*

$$\delta(v_1 - v_2 + \cdots - v_{2n+2} + \sum_{l \in T} u_l) e^{iVQV + iURU + iUSV} \tag{4.7}$$

where the v variables are the long or external variables of the rosette, counted with their signs, and the quadratic oscillations for these variables is

$$VQV = \sum_{0 \leqslant i < j \leqslant r} (-1)^{i+j+1} v_i \theta^{-1} v_j \tag{4.8}$$

We have now to analyze in detail this quadratic oscillation of the remaining long loop variables since it is essential to improve power counting. We can neglect the secondary oscillations URU and USV which imply short variables.

The second Filk reduction [8] further simplifies the rosette factor by erasing the loops of the rosette which do not cross any other loops or arch over external fields. It can be shown that the loops which disappear in this operation correspond to those long variables who do not appear in the quadratic form Q.

Using the remaining *oscillating factors* one can prove that non-planar graphs with genus larger than one or with more than one external face *do not diverge*.

The basic mechanism to improve the power counting of a single non-planar subgraph is the following:

$$\begin{aligned}
&\int dw_1 dw_2 e^{-M^{-2i_1} w_1^2 - M^{-2i_2} w_2^2 - i w_1 \theta^{-1} w_2 + w_1 . E_1(x,u) + w_2 E_2(x,u)} \\
&\quad = \int dw_1' dw_2' e^{-M^{-2i_1}(w_1')^2 - M^{-2i_2}(w_2')^2 + i w_1' \theta^{-1} w_2' + (u,x) Q(u,x)} \\
&\quad = K M^{4i_1} \int dw_2' e^{-(M^{2i_1} + M^{-2i_2})(w_2')^2} = K M^{4i_1} M^{-4i_2}.
\end{aligned} \tag{4.9}$$

In these equations we used for simplicity M^{-2i} instead of the correct but more complicated factor $(\Omega/4)\tanh(\alpha/2)$ (of course this does not change the argument) and we performed a unitary linear change of variables $w_1' = w_1 + \ell_1(x,u)$, $w_2' = w_2 + \ell_2(x,u)$ to compute the oscillating w_1' integral. The gain in (4.9) is M^{-8i_2}, which is the difference between M^{-4i_2} and the normal factor M^{4i_2} that the w_2 integral would have cost if we had done it with the regular $e^{-M^{-2i_2} w_2^2}$ factor for long variables. To maximize this gain we can assume $i_1 \leqslant i_2$.

This basic argument must then be generalized to each non-planar subgraph in the multiscale analysis, which is possible.

Finally it remains to consider the case of subgraphs which are planar orientable but with more than one external face. In that case there are no crossing loops in the rosette but there must be at least one loop line arching over a non-trivial subset of external legs (see e.g. line 6 in Figure 6). We have then a non-trivial integration over at least one external variable, called x, of at least one long loop variable called w. This "external" x variable without the oscillation improvement would be integrated with a test function of scale 1 (if it is a true external line of scale 1) or better (if it is a higher long loop variable)[5]. But we get now

$$\int dxdwe^{-M^{-2i}w^2 - iw\theta^{-1}x + w.E_1(x',u)} = KM^{4i}\int dxe^{-M^{+2i}x^2} = K', \tag{4.10}$$

so that a factor M^{4i} in the former bound becomes $\mathcal{O}(1)$ hence is improved by M^{-4i}.

In this way we can reduce the convergence of the multiscale analysis to the problem of renormalisation of planar two- and four-point subgraphs with a single external face, which we treat in the next section.

Remark that the power counting obtained in this way is still not optimal. To get the same level of precision as with the matrix base requires e.g. to display g independent improvements of the type (4.9) for a graph of genus g. This is doable but basically requires a reduction of the quadratic form Q for single-faced rosette (also called "hyperrosette") into g standard symplectic blocks through the so-called "third Filk move" introduced in [19]. We return to this question in Section 4.4.

4.3 Renormalisation

4.3.1 Four-point function. Consider the amplitude of a four-point graph G which in the multiscale expansion has all its internal scales higher than its four external scales.

The idea is that one should compare its amplitude to a similar amplitude with a "Moyal factor" $\exp\big(2\imath\theta^{-1}(x_1\wedge x_2 + x_3\wedge x_4)\big)\delta(\Delta)$ factorized in front, where $\Delta = x_1 - x_2 + x_3 - x_4$. But precisely because the graph is planar with a single external face we understand that the external positions x only couple to *short variables U* of the internal amplitudes through the global δ-function and the oscillations. Hence we can break this coupling by a systematic Taylor expansion to first order. This separates a piece proportional to "Moyal factor", then absorbed into the effective coupling constant, and a remainder which has at least one additional small factor which gives him improved power counting.

[5] Since the loop line arches over a non-trivial (i.e. neither full nor empty) subset of external legs of the rosette, the variable x cannot be the full combination of external variables in the "root" δ-function.

This is done by expressing the amplitude for a graph with $N = 4$, $g = 0$ and $B = 1$ as follows:

$$A(G)(x_1, x_2, x_3, x_4) = \int \exp\left(2\imath\theta^{-1}\,(x_1 \wedge x_2 + x_3 \wedge x_4)\right) \prod_{\ell \in T_k^i} du_\ell \; C_\ell(u_\ell, U_\ell, V_\ell)$$
$$\Big[\prod_{l \in G_k^i \;\; l \notin T} du_l dv_l C_l(u_l, v_l)\Big]\, e^{\imath URU + \imath USV} \tag{4.11}$$
$$\Big\{\delta(\Delta) + \int_0^1 dt\big[\mathfrak{U} \cdot \nabla\delta(\Delta + t\mathfrak{U}) + \delta(\Delta + t\mathfrak{U})[\imath XQU + \mathfrak{R}'(t)]\big]e^{\imath tXQU + \mathfrak{R}(t)}\Big\},$$

where $C_\ell(u_\ell, U_\ell, V_\ell)$ is the propagator taken at $X_\ell = 0$, $\mathfrak{U} = \sum_\ell u_\ell$ and $\mathfrak{R}(t)$ is a correcting term involving $\tanh \alpha_\ell [X.X + X.(U + V)]$.

The first term is of the initial $\int \mathrm{Tr}\, \phi \star \phi \star \phi \star \phi$ form. The rest no longer diverges, since the U and $\mathfrak{R}$ provide the necessary small factors.

4.3.2 Two-point function. Following the same strategy we have to Taylor-expand the coupling between external variables and U factors in two point planar graphs with a single external face to *third order* and some non-trivial symmetrization of the terms according to the two external arguments to cancel some odd contributions. The corresponding factorized relevant and marginal contributions can be then shown to give rise only to

- A mass counterterm,
- A wave function counterterm,
- An harmonic potential counterterm.

and the remainder has convergent power counting. This concludes the construction of the effective expansion in this direct space multiscale analysis.

Again the BPHZ theorem itself for the renormalised expansion follows by developing the counterterms still hidden in the effective couplings and its finiteness follows from the standard classification of forests. See however the remarks at the end of Section 3.2.2.

Since the bound (4.5) works for any $\Omega \neq 0$, an additional bonus of the x-space method is that it proves renormalisability of the model for any Ω in $]0, 1]$[6], whether the matrix method proved it only for Ω in $]0.5, 1]$.

4.3.3 The Langmann–Szabo–Zarembo model. This model is a four-dimensional theory of a Bosonic complex field defined by the action

$$S = \int \frac{1}{2}\bar\phi(-D^\mu D_\mu + \Omega^2 x^2)\phi + \lambda\bar\phi \star \phi \star \bar\phi \star \phi, \tag{4.12}$$

[6]The case Ω in $[1, +\infty[$ is irrelevant since it can be rewritten by LS duality as an equivalent model with Ω in $]0, 1]$.

where $D^\mu = \imath\partial_\mu + B_{\mu\nu}x^\nu$ is the covariant derivative in a magnetic field B.

The interaction $\bar\phi\star\phi\star\bar\phi\star\phi$ ensures that perturbation theory contains only orientable graphs. For $\Omega > 0$ the x-space propagator still decays as in the ordinary ϕ^4_4 case and the model has been shown renormalisable by an easy extension of the methods of the previous section [3].

However at $\Omega = 0$, there is no longer any harmonic potential in addition to the covariant derivatives and the bounds are lost. Models in this category are called "critical".

4.3.4 Critical models. Consider the x-kernel of the operator

$$H^{-1} = \left(p^2 + \Omega^2\tilde{x}^2 - 2\imath B\left(x^0p_1 - x^1p_0\right)\right)^{-1}, \tag{4.13}$$

$$H^{-1}(x,y) = \frac{\widetilde{\Omega}}{8\pi}\int_0^\infty \frac{dt}{\sinh(2\widetilde{\Omega}t)}\exp\Bigg(-\frac{\widetilde{\Omega}}{2}\frac{\cosh(2Bt)}{\sinh(2\widetilde{\Omega}t)}(x-y)^2 \tag{4.14}$$

$$-\frac{\widetilde{\Omega}}{2}\frac{\cosh(2\widetilde{\Omega}t)-\cosh(2Bt)}{\sinh(2\widetilde{\Omega}t)}(x^2+y^2) \tag{4.15}$$

$$+2\imath\widetilde{\Omega}\frac{\sinh(2Bt)}{\sinh(2\widetilde{\Omega}t)}x\wedge y\Bigg) \qquad \text{with } \widetilde{\Omega} = \frac{2\Omega}{\theta}. \tag{4.16}$$

The Gross–Neveu model or the critical Langmann–Szabo–Zarembo models correspond to the case $B = \widetilde{\Omega}$. In these models there is no longer any confining decay for the "long variables" but only an oscillation:

$$Q^{-1} = H^{-1} = \frac{\widetilde{\Omega}}{8\pi}\int_0^\infty \frac{dt}{\sinh(2\widetilde{\Omega}t)}\exp\left(-\frac{\widetilde{\Omega}}{2}\coth(2\widetilde{\Omega}t)(x-y)^2 + 2\imath\widetilde{\Omega}x\wedge y\right). \tag{4.17}$$

This kind of models are called critical. Their construction is more difficult, since sufficiently many oscillations must be proven independent before power counting can be established. The prototype paper which solved this problem is [21], which we briefly summarize now.

The main technical difficulty of the critical models is the absence of decreasing functions for the long v variables in the propagator replaced by an oscillation, see (4.17). Note that these decreasing functions are in principle created by integration over the u variables[7]:

$$\int du\, e^{-\frac{\widetilde{\Omega}}{2}\coth(2\widetilde{\Omega}t)u^2 + \imath u\wedge v} = K\tanh(2\widetilde{\Omega}t)\, e^{-k\tanh(2\widetilde{\Omega}t)v^2}. \tag{4.18}$$

But to perform all these Gaussian integrations for a general graph is a difficult task (see [42]) and is in fact not necessary for a BPHZ theorem. We can instead exploit the vertices and propagators oscillations to get rational decreasing functions in some linear combinations of the long v variables. The difficulty is then to prove that all

[7] In all the following we restrict ourselves to the dimension 2.

these linear combinations are independent and hence allow to integrate over all the v variables. To solve this problem we need the exact expression of the total oscillation in terms of the short and long variables. This consists in a generalization of the Filk's work [8]. This has been done in [21]. Once the oscillations are proven independent, one can just use the same arguments as in the Φ^4 case (see Section 4.2) to compute an upper bound for the power counting:

Lemma 4.4 (Power counting GN^2_Θ). *Let G a connected orientable graph. For all $\Omega \in [0,1)$, there exists $K \in \mathbb{R}_+$ such that its amputated amplitude A_G integrated over test functions is bounded by*

$$|A_G| \leqslant K^n M^{-\frac{1}{2}\omega(G)} \tag{4.19}$$

$$\text{with } \omega(G) = \begin{cases} N-4 & \text{if } N=2 \text{ or } N \geqslant 6 \text{ and } g=0, \\ & \text{if } N=4,\ g=0 \text{ and } B=1, \\ & \text{if } G \text{ is critical}, \\ N & \text{if } N=4,\ g=0,\ B=2 \text{ and } G \text{ non-critical}, \\ N+4 & \text{if } g \geqslant 1. \end{cases} \tag{4.20}$$

As in the non-commutative Φ^4 case, only the planar graphs are divergent. But the behaviour of the graphs with more than one broken face is different. Note that we already discussed such a feature in the matrix basis (see Section 3.4). In the multiscale framework, the Feynman diagrams are endowed with a scale attribution which gives each line a scale index. The only subgraphs we meet in this setting have all their internal scales higher than their external ones. Then a subgraph G of scale i is called *critical* if it has $N=4, g=0, B=2$ and that the two "external" points in the second broken face are only linked by a single line of scale $j<i$. The typical example is the graph of Figure 4 (a). In this case, the subgraph is logarithmically divergent whereas it is convergent in the Φ^4 model. Let us now show roughly how it happens in the case of Figure 4 (a) but now in x-space.

The same arguments as in the Φ^4 model prove that the integrations over the internal points of the graph 4 (a) lead to a logarithmical divergence which means that $A_{G^i} \simeq \mathcal{O}(1)$ in the multiscale framework. But remind that there is a remaining oscillation between a long variable of this graph and the external points in the second broken face of the form $v \wedge (x-y)$. But v is of order M^i which leads to a decreasing function implementing $x-y$ of order M^{-i}. If these points are true external ones, they are integrated over test functions of norm 1. Then thanks to the additional decreasing function for $x-y$ we gain a factor M^{-2i} which makes the graph convergent. But if x and y are linked by a single line of scale $j<i$ (as in Figure 4 (b)), instead of test functions we have a propagator between x and y. This one behaves like (see (4.17)):

$$C^j(x,y) \simeq M^j\, e^{-M^{2j}(x-y)^2 + \imath x \wedge y}. \tag{4.21}$$

The integration over $x-y$ instead of giving M^{-2j} gives M^{-2i} thanks to the oscillation $v \wedge (x-y)$. Then we have gained a good factor $M^{-2(i-j)}$. But the oscillation in the propagator $x \wedge y$ now gives $x + y \simeq M^{2i}$ instead of M^{2j} and the integration over $x + y$ cancels the preceding gain. The critical component of Figure 4 (a) is logarithmically divergent.

This kind of argument can be repeated and refined for more general graphs to prove that this problem appears only when the external points of the auxiliary broken faces are linked only by a *single* lower line [21]. This phenomenon can be seen as a mixing between scales. Indeed the power counting of a given subgraph now depends on the graphs at lower scales. This was not the case in the commutative realm. Fortunately this mixing does not prevent renormalisation. Note that whereas the critical subgraphs are not renormalisable by a vertex-like counterterm, they are regularised by the renormalisation of the two-point function at scale j. The proof of this point relies heavily on the fact that there is only one line of lower scale.

Let us conclude this section by mentioning the flows of the critical models. One very interesting feature of the non-commutative Φ^4 model is the boundedness of its flows and even the vanishing of its beta function for a special value of its bare parameters [22], [23], [24]. Note that its commutative counterpart (the usual ϕ^4 model on $\mathbb{R}^4$) is asymptotically free in the infrared and has then an unbounded flow. It turns out that the flow of the critical models are not regularized by the non-commutativity. The one-loop computation of the beta functions of the non-commutative Gross–Neveu model [43] shows that it is asymptotically free in the ultraviolet region as in the commutative case.

4.4 Non-commutative hyperbolic polynomials

Since the Mehler kernel is quadratic it is possible to explicitly compute the non-commutative analogues of topological or "Symanzik" polynomials.

In ordinary commutative field theory, Symanzik's polynomials are obtained after integration over internal position variables. The amplitude of an amputated graph G with external momenta p is, up to a normalization, in space-time dimension D:

$$A_G(p) = \delta\Big(\sum p\Big) \int_0^\infty \frac{e^{-V_G(p,\alpha)/U_G(\alpha)}}{U_G(\alpha)^{D/2}} \prod_l (e^{-m^2\alpha_l}\, d\alpha_l). \tag{4.22}$$

The first and second Symanzik polynomials U_G and V_G are

$$U_G = \sum_T \prod_{l \notin T} \alpha_l, \tag{4.23a}$$

$$V_G = \sum_{T_2} \prod_{l \notin T_2} \alpha_l \Big(\sum_{i \in E(T_2)} p_i\Big)^2, \tag{4.23b}$$

where the first sum is over spanning trees T of G and the second sum is over two trees T_2, i.e. forests separating the graph in exactly two connected components $E(T_2)$

and $F(T_2)$; the corresponding Euclidean invariant $\big(\sum_{i\in E(T_2)} p_i\big)^2$ is, by momentum conservation, also equal to $\big(\sum_{i\in F(T_2)} p_i\big)^2$.

Since the Mehler kernel is still quadratic in position space it is possible to also integrate explicitly all positions to reduce Feynman amplitudes of e.g. non-commutative ϕ^4_4 purely to parametric formulas, but of course the analogs of Symanzik polynomials are now hyperbolic polynomials which encode the richer information about ribbon graphs. The reference for these polynomials is [19], which treats the ordinary ϕ^4_4 case. In [42], these polynomials are also computed in the more complicated case of critical models.

Defining the antisymmetric matrix σ as

$$\sigma = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad \text{with} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.24}$$

the $\delta-$functions appearing in the vertex contribution can be rewritten as an integral over some new variables p_V. We refer to these variables as to *hypermomenta*. Note that one associates such a hypermomenta p_V to any vertex V *via* the relation

$$\begin{aligned} \delta(x_1^V - x_2^V + x_3^V - x_4^V) &= \int \frac{dp'_V}{(2\pi)^4} e^{ip'_V(x_1^V - x_2^V + x_3^V - x_4^V)} \\ &= \int \frac{dp_V}{(2\pi)^4} e^{p_V\sigma(x_1^V - x_2^V + x_3^V - x_4^V)} . \end{aligned} \tag{4.25}$$

Consider a particular ribbon graph G. Specializing to dimension 4 and choosing a particular root vertex $\bar{V}$ of the graph, one can write the Feynman amplitude for G in the condensed way

$$\mathcal{A}_G = \int \prod_\ell \left[\frac{1-t_\ell^2}{t_\ell}\right]^2 d\alpha_\ell \int dx dp e^{-\frac{\tilde\Omega}{2} XGX^t} \tag{4.26}$$

where $t_\ell = \tanh \frac{\alpha_\ell}{2}$, X summarizes all positions and hypermomenta and G is a certain quadratic form. If we call x_e and $p_{\bar{V}}$ the external variables we can decompose G according to an internal quadratic form Q, an external one M and a coupling part P so that

$$X = \begin{pmatrix} x_e & p_{\bar{V}} & u & v & p \end{pmatrix}, \quad G = \begin{pmatrix} M & P \\ P^t & Q \end{pmatrix}, \tag{4.27}$$

Performing the Gaussian integration over all internal variables one obtains

$$\mathcal{A}_G = \int \left[\frac{1-t^2}{t}\right]^2 d\alpha \frac{1}{\sqrt{\det Q}} e^{-\frac{\tilde\Omega}{2}\begin{pmatrix} x_e & \bar{p} \end{pmatrix}[M - PQ^{-1}P^t]\begin{pmatrix} x_e \\ \bar{p} \end{pmatrix}} . \tag{4.28}$$

This form allows to define the polynomials $HU_{G,\bar{v}}$ and $HV_{G,\bar{v}}$, analogs of the Symanzik polynomials U and V of the commutative case (see (4.22)). They are defined by

$$\mathcal{A}_{\bar{V}}(\{x_e\},\ p_{\bar{v}}) = K' \int_0^\infty \prod_l [d\alpha_l (1-t_l^2)^2] HU_{G,\bar{v}}(t)^{-2} e^{-\frac{HV_{G,\bar{v}}(t,x_e,p_{\bar{v}})}{HU_{G,\bar{v}}(t)}}. \quad (4.29)$$

They are polynomials in the set of variables t_ℓ ($\ell = 1, \dots, L$), the hyperbolic tangent of the half-angle of the parameters α_ℓ.

Using now (4.28) and (4.29) the polynomial $HU_{G,\bar{v}}$ writes

$$HU_{\bar{v}} = (\det Q)^{\frac{1}{4}} \prod_{\ell=1}^{L} t_\ell. \quad (4.30)$$

The main results ([19]) are as follows.

- The polynomials $HU_{G,\bar{v}}$ and $HV_{G,\bar{v}}$ have a strong positivity property. Roughly speaking they are sums of monomials with positive integer coefficients. This positive integer property comes from the fact that each such coefficient is the square of a Pfaffian with integer entries.

- Leading terms can be identified in a given "Hepp sector", at least for *orientable graphs*. A Hepp sector is a complete ordering of the t parameters. These leading terms which can be shown strictly positive in $HU_{G,\bar{v}}$ correspond to super-trees which are the disjoint union of a tree in the direct graph and a tree in the dual graph. Hypertrees in a graph with n vertices and F faces have therefore $n + F - 2$ lines. (Any connected graph has hypertrees, and under reduction of the hypertree, the graph becomes a hyperrosette.) Similarly one can identify "super-two-trees" $HV_{G,\bar{v}}$ which govern the leading behavior of $HV_{G,\bar{v}}$ in any Hepp sector.

From the second property, one can deduce the *exact power counting* of any orientable ribbon graph of the theory, just as in the matrix base.

Let us now borrow from [19] some examples of these hyperbolic polynomials. We put $s = (4\theta\Omega)^{-1}$. For the bubble graph of Figure 7:

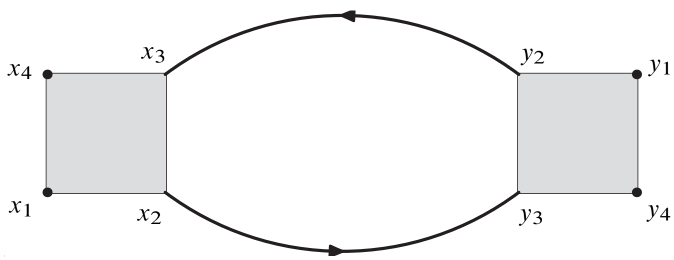


Figure 7. The bubble graph.

$$
\begin{aligned}
HU_{G,v} &= (1+4s^2)(t_1+t_2+t_1^2t_2+t_1t_2^2),\\
HV_{G,v} &= t_2^2\big[p_2+2s(x_4-x_1)\big]^2 + t_1t_2\big[2p_2^2+(1+16s^4)(x_1-x_4)^2\big],\\
&\qquad + t_1^2\big[p_2+2s(x_1-x_4)\big]^2.
\end{aligned} \tag{4.31}
$$

For the sunshine graph Figure 8:

$$
\begin{aligned}
HU_{G,v} &= \big[t_1t_2+t_1t_3+t_2t_3+t_1^2t_2t_3+t_1t_2^2t_3+t_1t_2t_3^2\big](1+8s^2+16s^4)\\
&\qquad + 16s^2(t_2^2+t_1^2t_3^2).
\end{aligned} \tag{4.32}
$$

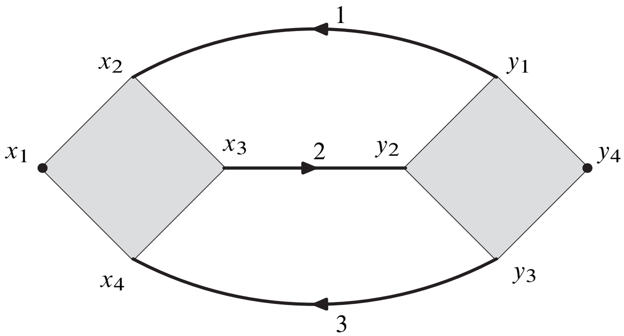


Figure 8. The sunshine graph.

For the non-planar sunshine graph (see Figure 9) we have

$$
\begin{aligned}
HU_{G,v} &= \big[t_1t_2+t_1t_3+t_2t_3+t_1^2t_2t_3+t_1t_2^2t_3+t_1t_2t_3^2\big](1+8s^2+16s^4)\\
&\qquad + 4s^2\big[1+t_1^2+t_2^2+t_1^2t_2^2+t_3^2+t_1^2t_3^2+t_2^2t_3^2+t_1^2t_2^2t_3^2\big].
\end{aligned} \tag{4.33}
$$

We note the improvement in the genus with respect to its planar counterparts.

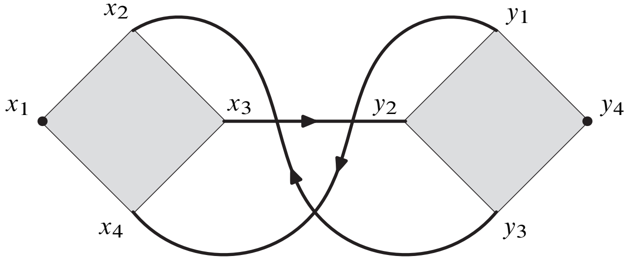


Figure 9. The non-planar sunshine graph.

For the broken bubble graph (see Figure 10) we have

$$
\begin{aligned}
HU_{G,v} &= (1+4s^2)(t_1+t_2+t_1^2t_2+t_1t_2^2),\\
HV_{G,v} &= t_2^2\Big[4s^2(x_1+y_2)^2+(p_2-2s(x_3+y_4))^2\Big]+t_1^2\Big[p_2+2s(x_3-y_4)\Big]^2,\\
&\quad +t_1t_2\Big[8s^2y_2^2+2(p_2-2sy_4)^2+(x_1+x_3)^2+16s^4(x_1-x_3)^2\Big]\\
&\quad +t_1^2t_2^24s^2(x_1-y_2)^2.
\end{aligned}
$$

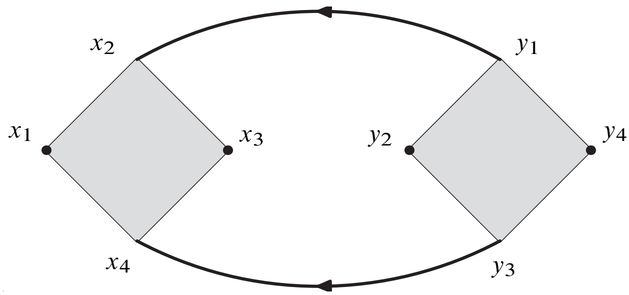


Figure 10. The broken bubble graph.

Note that $HU_{G,v}$ is identical to the one of the bubble with only one broken face. The power counting improvement comes from the broken face and can be seen only in $HV_{G,v}$.

Finally, for the half-eye graph (see Figure 11), we start by defining

$$
A_{24} = t_1t_3 + t_1t_3t_2^2 + t_1t_3t_4^2 + t_1t_3t_2^2t_4^2. \tag{4.34}
$$

The $HU_{G,v}$ polynomial with fixed hypermomentum corresponding to the vertex with

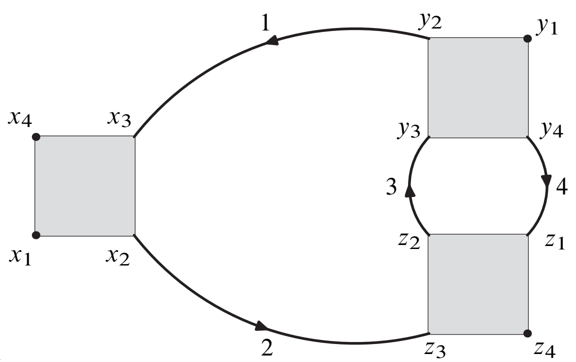


Figure 11. The half-eye graph.

two external legs is

$$
\begin{aligned}
HU_{G,v_1} = &(A_{24} + A_{14} + A_{23} + A_{13} + A_{12})(1 + 8s^2 + 16s^4) \\
&+ t_1t_2t_3t_4(8 + 16s^2 + 256s^4) + 4t_1t_2t_3^2 + 4t_1t_2t_4^2 \\
&+ 16s^2(t_3^2 + t_2^2t_4^2 + t_1^2t_4^2 + t_1^2t_2^2t_3^2) \\
&+ 64s^4(t_1t_2t_3^2 + t_1t_2t_4^2),
\end{aligned}
\tag{4.35}
$$

whereas with another fixed hypermomentum we get

$$
\begin{aligned}
HU_{G,v_2} = &(A_{24} + A_{14} + A_{23} + A_{13} + A_{12})(1 + 8s^2 + 16s^4) \\
&+ t_1t_2t_3t_4(4 + 32s^2 + 64s^4) + 32s^2t_1t_2t_3^2 + 32s^2t_1t_2t_4^2 \\
&+ 16s^2(t_3^2 + t_1^2t_4^2 + t_2^2t_4^2 + t_1^2t_2^3t_3^2).
\end{aligned}
\tag{4.36}
$$

Note that the leading terms are identical and the choice of the root perturbs only the non-leading ones. Moreover note the presence of the t_3^2 term. Its presence can be understood by the fact that in the sector $t_1, t_2, t_4 > t_3$ the subgraph formed by the lines $1, 2, 4$ has two broken faces. This is the sign of a power counting improvement due to the additional broken face in that sector. To exploit it, we have just to integrate over the variables of line 3 in that sector, using the second polynomial $HV_{G',v}$ for the triangle subgraph G' made of lines $1, 2, 4$.

In the *critical case* it is essential to introduce arrows upon the lines and to take them into account. The corresponding analysis together with many examples are given in [42].

4.5 Conclusion

Non-commutative QFT seemed initially to have non-renormalisable divergencies, due to UV/IR mixing. But following the Grosse–Wulkenhaar breakthrough, there has been recent rapid progress in our understanding of renormalisable QFT on Moyal spaces. We can already propose a preliminary classification of these models into different categories, according to the behavior of their propagators:

- Ordinary models at $0 < \Omega < 1$ such as ϕ^4_4 (which has non-orientable graphs) or $(\bar\phi\phi)^2$ models (which has none). Their propagator, roughly $(p^2 + \Omega^2\tilde{x}^2 + A)^{-1}$ is LS covariant and has good decay both in matrix space (3.11)–(3.14) and direct space (4.2). They have non-logarithmic mass divergencies and definitely require "vulcanization" i.e. the Ω term.

- "Supermodels", namely ordinary models but at $\Omega = 1$ in which the propagator is LS invariant. Their propagator is even better. In the matrix base it is diagonal, e.g. of the form $G_{m,n} = (m + n + A)^{-1}$, where A is a constant. The supermodels seem generically ultraviolet fixed points of the ordinary models, at which non-trivial Ward identities force the vanishing of the beta function. The flow of Ω to the $\Omega = 1$ fixed point is very fast (exponentially fast in RG steps).

- "Critical models" such as orientable versions of LSZ or Gross–Neveu (and presumably orientable gauge theories of various kind: Yang–Mills, Chern–Simons…). They may have only logarithmic divergencies and apparently no perturbative UV/IR mixing. However the vulcanized version still appears the most generic framework for their treatment. The propagator is then roughly $(p^2 + \Omega^2 \tilde{x}^2 + 2\Omega \tilde{x} \wedge p)^{-1}$. In matrix space this propagator shows definitely a weaker decay (3.38) than for the ordinary models, because of the presence of a non-trivial saddle point. In direct space the propagator no longer decays with respect to the long variables, but only oscillates. Nevertheless the main lesson is that in matrix space the weaker decay can still be used; and in x space the oscillations can never be completely killed by the vertices oscillations. Hence these models retain therefore essentially the power counting of the ordinary models, up to some nasty details concerning the four-point subgraphs with two external faces. Ultimately, thanks to a little conspiration in which the four-point subgraphs with two external faces are renormalised by the mass renormalisation, the critical models remain renormalisable. This is the main message of [21], [38].
- "Hypercritical models" which are of the previous type but at $\Omega = 1$. Their propagator in the matrix base is diagonal and depends only on one index m (e.g. always the left side of the ribbon). It is of the form $G_{m,n} = (m + A)^{-1}$. In x space the propagator oscillates in a way that often exactly compensates the vertices oscillations. These models have definitely worse power counting than in the ordinary case, with e.g. quadratically divergent four point-graphs (if sharp cut-offs are used). Nevertheless Ward identities can presumably still be used to show that they can still be renormalised. This probably requires a much larger conspiration to generalize the Ward identities of the supermodels.

Notice that the status of non-orientable critical theories is not yet clarified.

Parametric representation can be derived in the non-commutative case. It implies hyper-analogs of Symanzik polynomials which condense the information about the rich topological structure of a ribbon graph. Using this representation, dimensional regularization and dimensional renormalisation should extend to the non-commutative framework.

Remark that trees, which are the building blocks of the Symanzik polynomials, are also at the heart of (commutative) constructive theory, whose philosophy could be roughly summarized as "You shall use trees[8], but you shall *not* develop their loops or else you shall diverge". It is quite natural to conjecture that hypertrees, which are the natural non-commutative objects intrinsic to a ribbon graph, should play a key combinatoric role in the yet to develop non-commutative constructive field theory.

In conclusion we have barely started to scratch the world of renormalisable QFT on non-commutative spaces. The little we see through the narrow window now open is extremely tantalizing. There exists renormalisable NCQFTs eg ϕ^4 on $\mathbb{R}^4_\theta$, Gross–Neveu

[8]These trees may be either true trees of the graphs in the Fermionic case or trees associated to cluster or Mayer expansions in the Bosonic case, but this distinction is not essential.

on $\mathbb{R}^2_\theta$ and they seem to enjoy better properties than their commutative counterparts, for instance they no longer have Landau ghosts! Non-commutative non-relativistic field theories with a chemical potential seem the right formalism for a study ab initio of condensed matter in presence of a magnetic field, and in particular of the quantum Hall Effect. The correct scaling and RG theory of this effect presumably requires to build a very singular theory (of the hypercritical type) because of the huge degeneracy of the Landau levels. To understand this theory and the gauge theories on non-commutative spaces seem the most obvious challenges ahead of us.

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Mould expansions for the saddle-node and resurgence monomials

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Abstract. This article is an introduction to some aspects of Écalle's mould calculus, a powerful combinatorial tool which yields surprisingly explicit formulas for the normalising series attached to an analytic germ of singular vector field or of map. This is illustrated on the case of the saddle-node, a two-dimensional vector field which is formally conjugate to Euler's vector field $x^2\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}$, and for which the formal normalisation is shown to be resurgent in $1/x$. Resurgence monomials adapted to alien calculus are also described as another application of mould calculus.

Contents

1 Introduction

Mould calculus was developed by J. Écalle in relation with his resurgence theory almost thirty years ago ([3], [6], [7]). The primary goal of this text is to give an introduction to mould calculus, together with an exposition of the way it can be applied to a specific geometric problem pertaining to the theory of dynamical systems: the analytic classification of saddle-node singularities.

The treatment of this example was indicated in [4] in concise manner (see also [2]), but I found it useful to provide a self-contained presentation of mould calculus and detailed explanations for the saddle-node problem, in the same spirit as resurgence theory and alien calculus were presented in [14] together with the example of the analytic classification of tangent-to-identity transformations in complex dimension 1.

Basic facts from resurgence theory are also recalled in the course of the exposition, with the hope that this text will serve to a broad readership. I also included a section on the relation between the resurgent approach to the saddle-node problem and Martinet–Ramis's work [12].

The text consists of three parts.

A. Section 2 describes the problem of the normalisation of the saddle-node and Section 3 outlines its treatment by the method of mould-comould expansions.

B. The second part has an "algebraic" flavour: it is devoted to a systematic exposition of some features of mould algebras (Sections 4 and 5) and mould-comould expansions (Sections 6 and 7).

C. The third part is mainly concerned by the applications to resurgence theory of the previous results (Sections 8–11 show the consequences for the problem of the saddle-node and have an "analytic" flavour, Section 12 describes the construction of resurgence monomials which allow one to check the freeness of the algebra of alien derivations); other applications are also briefly alluded to in Section 13 (with a few words about arborification and multizetas).

All the ideas come from J. Écalle's articles and lectures. An effort has been made to provide full details, which occasionally may have resulted in original definitions, but they must be considered as auxiliary with respect to the overall theory. The details of the resurgence proofs which are given in Sections 8 and 10 are original, at least I did not see them in the literature previously.

A The saddle-node problem

2 The saddle-node and its formal normalisation

2.1. Let us consider a germ of complex analytic 2-dimensional vector field

$$X = x^2 \frac{\partial}{\partial x} + A(x,y)\frac{\partial}{\partial y}, \qquad A \in \mathbb{C}\{x,y\}, \tag{2.1}$$

for which we assume

$$A(0,y) = y, \quad \frac{\partial^2 A}{\partial x \partial y}(0,0) = 0. \tag{2.2}$$

Assumption (2.2) ensures that X is formally conjugate to the normal form

$$X_0 = x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \tag{2.3}$$

We shall be interested in the formal transformations which conjugate X and X_0.

2.2. This is the simplest case from the point of view of *formal* classification of saddle-node singularities of analytic differential equations.[1] Indeed, when a differential equation $B(x,y)\,\mathrm{d}y - A(x,y)\,\mathrm{d}x = 0$ is singular at the origin ($A(0,0) = B(0,0) = 0$) and its 1-jet has eigenvalues 0 and 1, it is always formally conjugate to one of the normal forms $x^{p+1}\,\mathrm{d}y - (1+\lambda x)y\,\mathrm{d}x = 0$ ($p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$) or $y\,\mathrm{d}x = 0$. What we call saddle-node singularity corresponds to the first case, and the normal form X_0 corresponds to $p = 1$ and $\lambda = 0$.

Moreover, a saddle-node singularity can always be analytically reduced to the form $x^{p+1}\,\mathrm{d}y - A(x,y)\,\mathrm{d}x = 0$ with $A(0,y) = y$ (this result goes back to Dulac – see [12], [13]), it is thus legitimate to consider vector fields of the form (2.1), which generate the same foliations (we restrict ourselves to $(p,\lambda) = (1,0)$ for the sake of simplicity).

The problem of the *analytic* classification of saddle-node singularities was solved in [12]. The resurgent approach to this problem is indicated in [4] and [3, Vol. 3] (see also [2]). The resurgent approach consists in analysing the divergence of the normalising transformation through alien calculus.

2.3. Normalising transformation means a formal diffeomorphism θ solution of the conjugacy equation

$$X = \theta_* X_0. \tag{2.4}$$

Due to the shape of X, one can find a unique formal solution of the form

$$\theta(x,y) = \big(x, \varphi(x,y)\big), \quad \varphi(x,y) = y + \sum_{n \geq 0} \varphi_n(x) y^n, \quad \varphi_n(x) \in x\mathbb{C}[[x]]. \tag{2.5}$$

The first step in the resurgent approach consists in proving that the formal series φ_n are resurgent with respect to the variable $z = -1/x$. We shall prove this fact by using Écalle's mould calculus (see Theorem 2 in Section 8 below).

The Euler series $\varphi_0(x) = -\sum_{n \geq 1}(n-1)!x^n$ appears in the case $A(x,y) = x + y$, for which the solution of the conjugacy equation is simply $\theta(x,y) = \big(x, y + \varphi_0(x)\big)$.

[1] A singular differential equation is essentially the same thing as a differential 1-form which vanishes at the origin. It defines a singular foliation, the leaves of which can also be obtained by integrating a singular vector field, but classifying singular foliations (or singular differential equations) is equivalent to classifying singular vector fields *up to time-change*. See e.g. [13].

2.4. Observe that $\theta(x,y) = \big(x, y + \sum_{n\geq 0} \varphi_n(x) y^n\big)$ is solution of the conjugacy equation if and only if

$$\widetilde{Y}(z,u) = u\,\mathrm{e}^z + \sum_{n\geq 0} u^n \mathrm{e}^{nz} \widetilde{\varphi}_n(z), \quad \widetilde{\varphi}_n(z) = \varphi_n(-1/z) \in z^{-1}\mathbb{C}[[z^{-1}]], \tag{2.6}$$

is solution of the differential equation

$$\partial_z \widetilde{Y} = A(-1/z, \widetilde{Y}) \tag{2.7}$$

associated with the vector field X. (Indeed, the first component of the flow of X is trivial and the second component is determined by solving (2.7); on the other hand, the flow of X_0 is trivial and, by plugging it into θ, one obtains the flow of X.)

The formal expansion $\widetilde{Y}(z,u)$ is called *formal integral* of the differential equation (2.7). One can obtain its components $\widetilde{\varphi}_n(z)$ (and, consequently, the formal series $\varphi_n(x)$ themselves) as solutions of ordinary differential equations, by expanding (2.7) in powers of u:

$$\frac{\mathrm{d}\widetilde{\varphi}_0}{\mathrm{d}z} = A\big(-1/z, \widetilde{\varphi}_0(z)\big), \tag{2.8}$$

$$\frac{\mathrm{d}\widetilde{\varphi}_n}{\mathrm{d}z} + n\widetilde{\varphi}_n(z) = \partial_y A\big(-1/z, \widetilde{\varphi}_0(z)\big)\widetilde{\varphi}_n(z) + \widetilde{\chi}_n(z), \tag{2.9}$$

with $\widetilde{\chi}_n$ inductively determined by $\widetilde{\varphi}_0, \dots, \widetilde{\varphi}_{n-1}$. Only the first equation is non-linear. One can prove the resurgence of the $\widetilde{\varphi}_n$'s by exploiting their property of being the unique solutions in $z^{-1}\mathbb{C}[[z^{-1}]]$ of these equations and devising a perturbative scheme to solve the first one,[2] but mould calculus is quite a different approach.

3 Mould-comould expansions for the saddle-node

3.1. The analytic vector fields X and X_0 can be viewed as derivations of the algebra $\mathbb{C}\{x,y\}$, but since we are interested in formal conjugacy, we now consider them as derivations of $\mathbb{C}[[x,y]]$. We shall first rephrase our problem as a problem about operators of this algebra.[3]

The commutative algebra $\mathcal{A} = \mathbb{C}[[x,y]]$ is also a local ring; as such, it is endowed with a metrizable topology, in which the powers of the maximal ideal $\mathfrak{M} = \{\, f \in \mathbb{C}[[x,y]] \mid f(0,0) = 0 \,\}$ form a system of neighbourhoods of 0, which we call Krull topology or topology of the formal convergence and which is complete (as a uniform structure).[4]

[2] See Section 2.1 of [14] for an illustration of this method on a non-linear difference equation.

[3] Our algebras will be, unless otherwise specified, associative unital algebras over $\mathbb{C}$ (possibly non-commutative). In this article, operator means endomorphism of the underlying vector space; thus an operator of $\mathcal{A} = \mathbb{C}[[x,y]]$ is an element of $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$. The space $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ has natural structures of ring and of $\mathcal{A}$-module, which are compatible in the sense that $f \in \mathcal{A} \mapsto f\,\mathrm{Id} \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ is a ring homomorphism.

Lemma 3.1. *The set of all continuous algebra homomorphisms of* $\mathbb{C}[[x,y]]$ *coincides with the set of all substitution operators, i.e. operators of the form* $f \mapsto f \circ \theta$ *with* $\theta \in \mathfrak{M} \times \mathfrak{M}$.

Proof. Any substitution operator is clearly a continuous algebra homomorphism of $\mathbb{C}[[x,y]]$. Conversely, let Θ be a continuous algebra homomorphism. The idea is that Θ will be determined by its action on the two generators of the maximal ideal, and setting $\theta = (\Theta x, \Theta y)$ we can identify Θ with the substitution operator $f \mapsto f \circ \theta$. We just need to check that Θx and Θy both belong to the maximal ideal, which is the case because, by continuity, $(\Theta x)^n = \Theta(x^n)$ and $(\Theta y)^n = \Theta(y^n)$ must tend to 0 as $n \to \infty$; one can then write any f as a convergent – for the Krull topology – series of monomials $\sum f_{m,n} x^m y^n$ and its image as the formally convergent series $\Theta f = \sum \Theta(f_{m,n} x^m y^n) = \sum f_{m,n} (\Theta x)^m (\Theta y)^n$. $\square$

A formal invertible transformation thus amounts to a continuous automorphism of $\mathbb{C}[[x,y]]$. Since the conjugacy equation (2.4) can be written

$$Xf = \left[X_0(f \circ \theta)\right] \circ \theta^{-1}, \quad f \in \mathbb{C}[[x,y]],$$

if we work at the level of the substitution operator, we are left with the problem of finding a continuous automorphism Θ of $\mathbb{C}[[x,y]]$ such that $\Theta(Xf) = X_0(\Theta f)$ for all f, i.e.

$$\Theta X = X_0 \Theta. \tag{3.1}$$

3.2. The idea is to construct a solution to (3.1) from the "building blocks" of X. Let us use the Taylor expansion

$$A(x,y) = y + \sum_{n \in \mathcal{N}} a_n(x) y^{n+1}, \quad \mathcal{N} = \{ n \in \mathbb{Z} \mid n \geq -1 \} \tag{3.2}$$

to write

$$X = X_0 + \sum_{n \in \mathcal{N}} a_n(x) B_n, \quad B_n = y^{n+1} \frac{\partial}{\partial y}, \tag{3.3}$$

$$a_n(x) \in x\mathbb{C}\{x\}, \quad a_0(x) \in x^2 \mathbb{C}\{x\} \tag{3.4}$$

(thus incorporating the information from (2.2)). The series in (3.3) must be interpreted as a simply convergent series of operators of $\mathbb{C}[[x,y]]$ (the series $\sum a_n B_n f$ is formally convergent for any $f \in \mathbb{C}[[x,y]]$).

Let us introduce the differential operators

$$\boldsymbol{B}_{\emptyset} = \mathrm{Id}, \quad \boldsymbol{B}_{\omega_1,\dots,\omega_r} = B_{\omega_r} \cdots B_{\omega_1} \tag{3.5}$$

[4] This is also called the $\mathfrak{M}$-adic topology, or the (x,y)-adic topology. Beware that $\mathbb{C}[[x,y]]$ is a topological algebra only if we put the discrete topology on $\mathbb{C}$.

for $\omega_1, \dots, \omega_r \in \mathcal{N}$. We shall look for an automorphism Θ solution of (3.1) in the form

$$\Theta = \sum_{r \geq 0} \sum_{\omega_1, \dots, \omega_r \in \mathcal{N}} \mathcal{V}^{\omega_1, \dots, \omega_r}(x) \boldsymbol{B}_{\omega_1, \dots, \omega_r}, \tag{3.6}$$

with the convention that the only term with $r = 0$ is $\mathcal{V}^{\emptyset} \boldsymbol{B}_{\emptyset}$, with $\mathcal{V}^{\emptyset} = 1$, and with coefficients $\mathcal{V}^{\omega_1, \dots, \omega_r}(x) \in x\mathbb{C}[[x]]$ to be determined from the data $\{a_n, \, n \in \mathcal{N}\}$ in such a way that

(i) the expression (3.6) is a formally convergent series of operators of $\mathbb{C}[[x, y]]$ and defines an operator Θ which is continuous for the Krull topology,

(ii) the operator Θ is an algebra automorphism,

(iii) the operator Θ satisfies the conjugacy equation (3.1).

In Écalle's terminology, the collection of operators $\{\boldsymbol{B}_{\omega_1, \dots, \omega_r}\}$ is a typical example of *comould*; any collection of coefficients $\{\mathcal{V}^{\omega_1, \dots, \omega_r}\}$ is a *mould* (here with values in $\mathbb{C}[[x]]$, but other algebras may be used); a formally convergent series of the form (3.6) is a *mould-comould expansion*, often abbreviated as

$$\Theta = \sum \mathcal{V}^{\bullet} \boldsymbol{B}_{\bullet}$$

(we shall clarify later what "formally convergent" means for such multiply-indexed series of operators).

3.3. Let us indicate right now the formulas for the problem of the saddle-node (2.1):

Lemma 3.2. *The equations*

$$\mathcal{V}^{\emptyset} = 1,$$

$$\left(x^2 \frac{\mathrm{d}}{\mathrm{d}x} + \omega_1 + \dots + \omega_r \right) \mathcal{V}^{\omega_1, \dots, \omega_r} = a_{\omega_1} \mathcal{V}^{\omega_2, \dots, \omega_r}, \quad \omega_1, \dots, \omega_r \in \mathcal{N} \tag{3.7}$$

inductively determine a unique collection of formal series $\mathcal{V}^{\omega_1, \dots, \omega_r} \in x\mathbb{C}[[x]]$ *for* $r \geq 1$. *Moreover,*

$$\mathcal{V}^{\omega_1, \dots, \omega_r} \in x^{\lceil r/2 \rceil} \mathbb{C}[[x]], \tag{3.8}$$

where $\lceil s \rceil$ *denotes, for any* $s \in \mathbb{R}$, *the least integer not smaller than* s.

Proof. Let ν denote the valuation in $\mathbb{C}[[x]]$: $\nu(\sum c_m x^m) = \min\{ m \mid c_m \neq 0 \} \in \mathbb{N}$ for a non-zero formal series and $\nu(0) = \infty$.

Since $\partial = x^2 \frac{\mathrm{d}}{\mathrm{d}x}$ increases valuation by at least one unit, $\partial + \mu$ is invertible for any $\mu \in \mathbb{C}^*$ and the inverse operator

$$(\partial + \mu)^{-1} = \sum_{r \geq 0} \mu^{-r-1} (-\partial)^r \tag{3.9}$$

(formally convergent series of operators) leaves $x\mathbb{C}[[x]]$ invariant. On the other hand, we *define* $\partial^{-1} \colon x^2\mathbb{C}[[x]] \to x\mathbb{C}[[x]]$ by the formula $\partial^{-1}\varphi(x) = \int_0^x \left(t^{-2} \varphi(t) \right) \mathrm{d}t$, so that $\psi = \partial^{-1}\varphi$ is the unique solution in $x\mathbb{C}[[x]]$ of the equation $\partial\psi = \varphi$ whenever $\varphi \in x^2\mathbb{C}[[x]]$.

For $r = 1$, equation (3.7) has a unique solution $\mathcal{V}^{\omega_1}$ in $x\mathbb{C}[[x]]$, because the right-hand side is a_{ω_1}, element of $x\mathbb{C}[[x]]$, and even of $x^2\mathbb{C}[[x]]$ when $\omega_1 = 0$. By induction, for $r \geq 2$, we get a right-hand side in $x^2\mathbb{C}[[x]]$ and a unique solution $\mathcal{V}^{\boldsymbol{\omega}}$ in $x\mathbb{C}[[x]]$ for $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r) \in \mathcal{N}^r$. Moreover, with the notation ${}^{\backprime}\boldsymbol{\omega} = (\omega_2, \ldots, \omega_r)$, we have

$$\nu(\mathcal{V}^{\boldsymbol{\omega}}) \geq \alpha^{\boldsymbol{\omega}} + \nu(\mathcal{V}^{{}^{\backprime}\boldsymbol{\omega}}), \quad \text{with } \alpha^{\boldsymbol{\omega}} = \begin{cases} 0 & \text{if } \omega_1 + \cdots + \omega_r = 0 \text{ and } \omega_1 \neq 0, \\ 1 & \text{if } \omega_1 + \cdots + \omega_r \neq 0 \text{ or } \omega_1 = 0. \end{cases}$$

Thus $\nu(\mathcal{V}^{\boldsymbol{\omega}}) \geq \operatorname{card} \mathcal{R}^{\boldsymbol{\omega}}$, with $\mathcal{R}^{\boldsymbol{\omega}} = \{\, i \in [1, r] \mid \omega_i + \cdots + \omega_r \neq 0 \text{ or } \omega_i = 0 \,\}$ for $r \geq 1$.

Let us check that $\operatorname{card} \mathcal{R}^{\boldsymbol{\omega}} \geq \lceil r/2 \rceil$. This stems from the fact that if $i \notin \mathcal{R}^{\boldsymbol{\omega}}$, $i \geq 2$, then $i - 1 \in \mathcal{R}^{\boldsymbol{\omega}}$ (indeed, in that case $\omega_{i-1} + \cdots + \omega_r = \omega_{i-1}$), and that $\mathcal{R}^{\boldsymbol{\omega}}$ has at least one element, namely r. The inequality is thus true for $r = 1$ or 2; by induction, if $r \geq 3$, then $\mathcal{R}^{\boldsymbol{\omega}} \cap [3, r] = \mathcal{R}^{{}^{\backprime\backprime}\boldsymbol{\omega}}$ with ${}^{\backprime\backprime}\boldsymbol{\omega} = (\omega_3, \ldots, \omega_r)$ and either $2 \in \mathcal{R}^{\boldsymbol{\omega}}$, or $2 \notin \mathcal{R}^{\boldsymbol{\omega}}$ and $1 \in \mathcal{R}^{\boldsymbol{\omega}}$, thus $\operatorname{card} \mathcal{R}^{\boldsymbol{\omega}} \geq 1 + \operatorname{card} \mathcal{R}^{{}^{\backprime\backprime}\boldsymbol{\omega}}$. □

3.4. To give a definition of formally summable families of operators adapted to our needs, we shall consider our operators as elements of a topological ring of a certain kind and make use of the Cauchy criterium for summable families.

Definition 3.1. Given a ring $\mathcal{E}$ (possibly non-commutative), we call pseudovaluation any map $\operatorname{val}\colon \mathcal{E} \to \mathbb{Z} \cup \{\infty\}$ satisfying, for any $\Theta, \Theta_1, \Theta_2 \in \mathcal{E}$,

- $\operatorname{val}(\Theta) = \infty$ iff $\Theta = 0$,
- $\operatorname{val}(\Theta_1 - \Theta_2) \geq \min\{\operatorname{val}(\Theta_1), \operatorname{val}(\Theta_2)\}$,
- $\operatorname{val}(\Theta_1\Theta_2) \geq \operatorname{val}(\Theta_1) + \operatorname{val}(\Theta_2)$,

The formula $d_{\mathrm{val}}(\Theta_1, \Theta_2) = 2^{-\operatorname{val}(\Theta_2 - \Theta_1)}$ then defines a distance, for which $\mathcal{E}$ is a topological ring. We call $(\mathcal{E}, \operatorname{val})$ a complete pseudovaluation ring if the distance d_{val} is complete.

We use the word pseudovaluation rather than valuation because $\mathcal{E}$ is not assumed to be an integral domain, and we dot impose equality in the third property. The distance d_{val} is ultrametric, translation-invariant, and it satisfies $d_{\mathrm{val}}(0, \Theta_1\Theta_2) \leq d_{\mathrm{val}}(0, \Theta_1)\, d_{\mathrm{val}}(0, \Theta_2)$.

Let us denote by 1 the unit of $\mathcal{E}$. Giving a pseudovaluation on $\mathcal{E}$ such that $\operatorname{val}(1) = 0$ is equivalent to giving a filtration $(\mathcal{E}_\delta)_{\delta \in \mathbb{Z}}$ that is compatible with its ring structure (i.e. a sequence if additive subgroups such that $1 \in \mathcal{E}_0$, $\mathcal{E}_{\delta+1} \subset \mathcal{E}_\delta$ and $\mathcal{E}_\delta\mathcal{E}_{\delta'} \subset \mathcal{E}_{\delta+\delta'}$ for all $\delta, \delta' \in \mathbb{Z}$), exhaustive ($\bigcup \mathcal{E}_\delta = \mathcal{E}$) and separated ($\bigcap \mathcal{E}_\delta = \{0\}$). Indeed, the order function val associated with the filtration, defined by $\operatorname{val}(\Theta) = \sup\{\, \delta \in \mathbb{Z} \mid \Theta \in \mathcal{E}_\delta \,\}$, is then a pseudovaluation; conversely, one can set $\mathcal{E}_\delta = \{\, \Theta \in \mathcal{E} \mid \operatorname{val}(\Theta) \geq \delta \,\}$.

Definition 3.2. Let $(\mathcal{E}, \mathrm{val})$ be a complete pseudovaluation ring. Given a set I, a family $(\Theta_i)_{i\in I}$ in $\mathcal{E}$ is said to be formally summable if, for any $\delta \in \mathbb{Z}$, the set $\{ i \in I \mid \mathrm{val}\,(\Theta_i) \le \delta \}$ is finite (the support of the family is thus countable, if not I itself).

One can then check that, for any exhaustion $(I_k)_{k\in\mathbb{N}}$ by finite sets of the support of the family, the sequence $\sum_{i\in I_k} \Theta_i$ is a Cauchy sequence for $\mathrm{d}_{\mathrm{val}}$, and that the limit does not depend on the chosen exhaustion; the common limit is then denoted $\sum_{i\in I} \Theta_i$. Observe that there must exist $\delta_* \in \mathbb{Z}$ such that $\mathrm{val}\,(\Theta_i) \ge \delta_*$ for all $i \in I$.

3.5. We apply this to operators of $\mathcal{A} = \mathbb{C}[[x,y]]$ as follows. The Krull topology of $\mathcal{A}$ can be defined with the help of the monomial valuation

$$\nu_4(f) = \min\{ 4m + n \mid f_{m,n} \neq 0 \} \quad \text{for } f = \sum f_{m,n} x^m y^n \neq 0, \quad \nu_4(0) = \infty.$$

Indeed, for any sequence $(f_k)_{k\in\mathbb{N}}$ of $\mathcal{A}$,

$$f_k \xrightarrow[k\to\infty]{} 0 \iff \sum_{k\in\mathbb{N}} f_k \text{ formally convergent} \iff \nu_4(f_k) \xrightarrow[k\to\infty]{} \infty.$$

In particular, $(\mathbb{C}[[x,y]], \nu_4)$ is a complete pseudovaluation ring.

Suppose more generally that $(\mathcal{A}, \nu)$ is any complete pseudovaluation ring such that $\mathcal{A}$ is also an algebra. Corresponding to the filtration $\mathcal{A}_p = \{ f \in \mathcal{A} \mid \nu(f) \ge p \}$, $p \in \mathbb{Z}$, there is a filtration of $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$:

$$\mathcal{E}_\delta = \{ \Theta \in \mathrm{End}_{\mathbb{C}}(\mathcal{A}) \mid \Theta(\mathcal{A}_p) \subset \mathcal{A}_{p+\delta} \text{ for each } p \}, \quad \delta \in \mathbb{Z}.$$

Definition 3.3. Let $\delta \in \mathbb{Z}$. An element Θ of $\mathcal{E}_\delta$ is said to be an "operator of valuation $\ge \delta$". We then define $\mathrm{val}_\nu\,(\Theta) \in \mathbb{Z} \cup \{\infty\}$, the "valuation of Θ", as the largest δ_0 such that Θ has valuation $\ge \delta_0$; this number is infinite only for $\Theta = 0$.

Denote by $\mathcal{E}$ the union $\bigcup \mathcal{E}_\delta$ over all $\delta \in \mathbb{Z}$: these are the operators of $\mathcal{A}$ "having a valuation" (with respect to ν), i.e.

$$\mathcal{E} = \{ \Theta \in \mathrm{End}_{\mathbb{C}}(\mathcal{A}) \mid \mathrm{val}_\nu\,(\Theta) = \inf_{f\in\mathcal{A}\setminus\{0\}}\{\nu(\Theta f) - \nu(f)\} > -\infty \}.$$

They clearly are continuous for the topology induced by ν on $\mathcal{A}$; they form a subalgebra of the algebra of all continuous operators[5] and $(\mathcal{E}, \mathrm{val}_\nu)$ is a complete pseudovaluation ring. For any formally summable family $(\Theta_i)_{i\in I}$ of sum Θ in $\mathcal{E}$ and $f \in \mathcal{A}$, the family $(\Theta_i f)_{i\in I}$ is summable in the topological ring $\mathcal{A}$, with sum Θf.

Lemma 3.3. *With the notation of formula (3.5) and Lemma 3.2, the family $(\mathcal{V}^{\omega_1,\dots,\omega_r} \boldsymbol{B}_{\omega_1,\dots,\omega_r})_{r\ge1,\,\omega_1,\dots,\omega_r\in\mathcal{N}}$ is formally summable in the algebra of operators of $\mathbb{C}[[x,y]]$ having a valuation with respect to ν_4. In particular the resulting*

[5] Not all continuous operators of $\mathcal{A}$ belong to $\mathcal{E}$: think of the operator of $\mathbb{C}[[y]]$ which maps y^m to $y^{m/2}$ if m is even and to $\mathbf{0}$ if m is odd.

operator Θ *is continuous for the Krull topology. Similarly, the formula*

$$\not{\mathcal{V}}^{\omega_1,\dots,\omega_r} = (-1)^r \mathcal{V}^{\omega_r,\dots,\omega_1} \tag{3.10}$$

gives rise to a formally summable family $(\not{\mathcal{V}}^{\omega_1,\dots,\omega_r} \boldsymbol{B}_{\omega_1,\dots,\omega_r})_{r\geq 1,\,\omega_1,\dots,\omega_r\in\mathcal{N}}$.

Proof. Clearly $\nu_4(B_n f) \geq \nu_4(f) + n$ and, by induction,

$$\nu_4(\boldsymbol{B}_{\omega_1,\dots,\omega_r} f) \geq \nu_4(f) + \omega_1 + \dots + \omega_r.$$

As a consequence of (3.8),

$$\nu_4(\mathcal{V}^{\omega_1,\dots,\omega_r} \boldsymbol{B}_{\omega_1,\dots,\omega_r} f) \geq \nu_4(f) + \omega_1 + \dots + \omega_r + 2r, \quad \omega_1,\dots,\omega_r \in \mathcal{N}.$$

Hence, with the above notations, each $\mathcal{V}^{\omega_1,\dots,\omega_r} \boldsymbol{B}_{\omega_1,\dots,\omega_r}$ is an element $\mathcal{E}$ with valuation $\geq \omega_1 + \dots + \omega_r + 2r$, and the same thing holds for each $\not{\mathcal{V}}^{\omega_1,\dots,\omega_r} \boldsymbol{B}_{\omega_1,\dots,\omega_r}$.

The ω_i's may be negative but they are always ≥ -1, thus $\omega_1 + \dots + \omega_r + r \geq 0$. Therefore, for any $\delta > 0$, the condition $\omega_1 + \dots + \omega_r + 2r \leq \delta$ implies $r \leq \delta$ and $\sum(\omega_i + 1) = \omega_1 + \dots + \omega_r + r \leq \delta$. Since this condition is fulfilled only a finite number of times, the conclusion follows. □

3.6. Here is the key statement, the proof of which will be spread over Sections 4–7:

Theorem 1. *The continuous operator* $\Theta = \sum \mathcal{V}^\bullet \boldsymbol{B}_\bullet$ *defined by Lemmas 3.2 and 3.3 is an algebra automorphism of* $\mathbb{C}[[x,y]]$ *which satisfies the conjugacy equation* (3.1). *The inverse operator is* $\sum \not{\mathcal{V}}^\bullet \boldsymbol{B}_\bullet$.

Observe that $\Theta x = x$, thus Θ is must be the substitution operator for a formal transformation of the form $\theta(x,y) = \big(x, \varphi(x,y)\big)$, with $\varphi = \Theta y$, in accordance with (2.5). An easy induction yields

$$\boldsymbol{B}_{\boldsymbol{\omega}} y = \beta_{\boldsymbol{\omega}} y^{\omega_1+\dots+\omega_r+1}, \quad \boldsymbol{\omega} \in \mathcal{N}^r,\ r \geq 1, \tag{3.11}$$

with $\beta_{\boldsymbol{\omega}} = 1$ if $r = 1$, $\beta_{\boldsymbol{\omega}} = (\omega_1+1)(\omega_1+\omega_2+1)\cdots(\omega_1+\dots+\omega_{r-1}+1)$ if $r \geq 2$. We have $\beta_{\boldsymbol{\omega}} = 0$ whenever $\omega_1 + \dots + \omega_r \leq -2$ (since (3.11) holds a priori in the fraction field $\mathbb{C}((y))$ but $\boldsymbol{B}_{\boldsymbol{\omega}} y$ belongs to $\mathbb{C}[[y]]$), hence

$$\theta(x,y) = \big(x, \varphi(x,y)\big),$$
$$\varphi(x,y) = y + \sum_{n\geq 0} \varphi_n(x) y^n, \quad \varphi_n = \sum_{\substack{r\geq 1,\,\boldsymbol{\omega}\in\mathcal{N}^r\\ \omega_1+\dots+\omega_r+1=n}} \beta_{\boldsymbol{\omega}} \mathcal{V}^{\boldsymbol{\omega}} \tag{3.12}$$

(in the series giving φ_n, there are only finitely many terms for each r, (3.8) thus yields its formal convergence in $x\mathbb{C}[[x]]$).

Similarly, $\Theta^{-1} = \sum \not{\mathcal{V}}^\bullet \boldsymbol{B}_\bullet$ is the substitution operator of a formal transformation $(x,y) \mapsto \big(x, \psi(x,y)\big)$, which is nothing but θ^{-1}, and

$$\psi(x,y) = \Theta^{-1} y = y + \sum_{n\geq 0} \psi_n(x) y^n, \tag{3.13}$$

where each coefficient can be represented as a formally convergent series $\psi_n = \sum_{\omega_1+\cdots+\omega_r+1=n} \beta_{\boldsymbol{\omega}} \mathcal{V}^{\boldsymbol{\omega}}$.

See Lemma 8.6 on p. 128 for formulas relating directly the φ_n's and the ψ_n's.

Remark 3.1. The $\mathcal{V}^{\boldsymbol{\omega}}$'s are generically divergent with at most Gevrey-1 growth of the coefficients, as can be expected from formula (3.9); for instance, for $\omega_1 \neq 0$, we get $\mathcal{V}^{\omega_1}(x) = \sum \omega_1^{-r-1} \left(-x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right)^r a_{\omega_1}$ which is generically divergent because the repeated differentiations are not compensated by the division by any factorial-like expression. This divergence is easily studied through formal Borel transform with respect to $z = -1/x$, which is the starting point of the resurgent analysis of the saddle-node – see Section 8. We shall see in Section 9 why the $\mathcal{V}^{\boldsymbol{\omega}}$'s can be called "resurgence monomials".

The proof of Theorem 1 will follow easily from the general notions introduced in the next sections.

B The formalism of moulds

4 The algebra of moulds

4.1. In this section and the next three ones, we assume that we are given a non-empty set Ω and a commutative $\mathbb{C}$-algebra $\boldsymbol{A}$, the unit of which is denoted 1. In the previous section, the roles of Ω and $\boldsymbol{A}$ were played by $\mathcal{N}$ and $\mathbb{C}[[x]]$.

It is sometimes convenient to have a commutative semigroup structure on Ω; then we would rather take $\Omega = \mathbb{Z}$ in the previous section and consider that the mould $\{\mathcal{V}^{\omega_1,\dots,\omega_r}\}$ was defined on $\mathbb{Z}$ but supported on $\mathcal{N}$ (i.e. we extend it by 0 whenever one of the ω_i's is ≤ -2).

We consider Ω as an alphabet and denote by $\Omega^\bullet$ the free monoid of *words*: a word is any finite sequence of letters, $\boldsymbol{\omega} = (\omega_1,\dots,\omega_r)$ with $\omega_1,\dots,\omega_r \in \Omega$; its *length* $r = r(\boldsymbol{\omega})$ can be any non-negative integer. The only word of zero length is the empty word, denoted $\emptyset$, which is the unit of *concatenation*, the monoid law $(\boldsymbol{\omega},\boldsymbol{\eta}) \mapsto \boldsymbol{\omega} \raisebox{.2ex}{$\scriptscriptstyle\blacksquare$} \boldsymbol{\eta}$ defined by

$$(\omega_1,\dots,\omega_r) \raisebox{.2ex}{$\scriptscriptstyle\blacksquare$} (\eta_1,\dots,\eta_s) = (\omega_1,\dots,\omega_r,\eta_1,\dots,\eta_s)$$

for non-empty words.

As previously alluded to, a *mould on* Ω *with values in* $\boldsymbol{A}$ is nothing but a map $\Omega^\bullet \to \boldsymbol{A}$. It is customary to denote the value of the mould on a word $\boldsymbol{\omega}$ by affixing $\boldsymbol{\omega}$ as an upper index to the symbol representing the mould, and to refer to the mould itself by using $\bullet$ as upper index. Hence $\mathcal{V}^\bullet$ is the mould, the value of which at $\boldsymbol{\omega}$ is denoted $\mathcal{V}^{\boldsymbol{\omega}}$.

A mould with values in $\mathbb{C}$ is called a *scalar mould*.

4.2. Being the set of all maps from a set to the ring $\boldsymbol{A}$, the set of moulds $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ has a natural structure of $\boldsymbol{A}$-module: addition and ring multiplication are defined component-wise (for instance, if $\mu \in \boldsymbol{A}$ and $M^\bullet \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$, the mould $N^\bullet = \mu M^\bullet$ is defined by $N^{\boldsymbol{\omega}} = \mu M^{\boldsymbol{\omega}}$ for all $\boldsymbol{\omega} \in \Omega^\bullet$).

The ring structure of $\boldsymbol{A}$ together with the monoid structure of $\Omega^\bullet$ also give rise to a *multiplication of moulds*, thus defined:

$$P^\bullet = M^\bullet \times N^\bullet\colon\ \boldsymbol{\omega} \mapsto P^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}=\boldsymbol{\omega}^1 \bullet \boldsymbol{\omega}^2} M^{\boldsymbol{\omega}^1} N^{\boldsymbol{\omega}^2}, \tag{4.1}$$

with summation over the $r(\boldsymbol{\omega}) + 1$ decompositions of $\boldsymbol{\omega}$ into two words (including $\boldsymbol{\omega}^1$ or $\boldsymbol{\omega}^2 = \emptyset$). Mould multiplication is associative but not commutative (except if Ω has only one element). We get a ring structure on $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$, with unit

$$1^\bullet\colon\ \boldsymbol{\omega} \mapsto 1^{\boldsymbol{\omega}} = \begin{cases} 1 & \text{if } \boldsymbol{\omega} = \emptyset, \\ 0 & \text{if } \boldsymbol{\omega} \neq \emptyset. \end{cases}$$

One can check that a mould $M^\bullet$ is invertible if and only if $M^\emptyset$ is invertible in $\boldsymbol{A}$ (see below).

One must in fact regard $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ as an $\boldsymbol{A}$-algebra, i.e. its module structure and ring structure are compatible: $\mu \in \boldsymbol{A} \mapsto \mu\, 1^\bullet \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ is indeed a ring homomorphism, the image of which lies in the center of the ring of moulds. The reader familiar with Bourbaki's *Elements of mathematics* will have recognized in $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ the large algebra (over $\boldsymbol{A}$) of the monoid $\Omega^\bullet$ (*Algèbre*, chap. III, §2, n°10). Other authors use the notation $\boldsymbol{A}\langle\langle\Omega\rangle\rangle$ or $\boldsymbol{A}[[\mathrm{T}^\Omega]]$ to denote this $\boldsymbol{A}$-algebra, viewing it as the completion of the free $\boldsymbol{A}$-algebra over Ω for the pseudovaluation ord defined below. The originality of moulds lies in the way they are used:

- the shuffling operation available in the free monoid $\Omega^\bullet$ will lead us in Section 5 to single out specific classes of moulds, enjoying certain symmetry or antisymmetry properties of fundamental importance (and this is only a small amount of all the structures used by Écalle in wide-ranging contexts);
- we shall see in Sections 6 and 7 how to contract moulds into “comoulds” (and this yields non-trivial results in the local study of analytic dynamical systems);
- the extra structure of commutative semigroup on Ω will allow us to define another operation, the “composition” of moulds (see below).

There is a pseudovaluation ord$\colon \mathcal{M}^\bullet(\Omega, \boldsymbol{A}) \to \mathbb{N} \cup \{\infty\}$, which we call “order”: we say that a mould $M^\bullet$ has order $\geq s$ if $M^{\boldsymbol{\omega}} = 0$ whenever $r(\boldsymbol{\omega}) < s$, and ord $(M^\bullet)$ is the largest such s. This way, we get a complete pseudovaluation ring $\big(\mathcal{M}^\bullet(\Omega, \boldsymbol{A}), \text{ord}\big)$. In fact, if $\boldsymbol{A}$ is an integral domain (as is the case of $\mathbb{C}[[x]]$), then $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ is an integral domain and ord is a valuation.

4.3. It is easy to construct “mould derivations”, i.e. $\mathbb{C}$-linear operators D of $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ such that $D(M^\bullet \times N^\bullet) = (DM^\bullet) \times N^\bullet + M^\bullet \times DN^\bullet$.

For instance, for any function $\varphi\colon \Omega \to \boldsymbol{A}$, the formula

$$D_\varphi M^{\boldsymbol{\omega}} = \begin{cases} 0 & \text{if } \boldsymbol{\omega} = \emptyset, \\ \big(\varphi(\omega_1) + \cdots + \varphi(\omega_r)\big) M^{\boldsymbol{\omega}} & \text{if } \boldsymbol{\omega} = (\omega_1, \ldots, \omega_r) \end{cases}$$

defines a mould derivation D_φ. With $\varphi \equiv 1$, we get $DM^{\boldsymbol{\omega}} = r(\boldsymbol{\omega}) M^{\boldsymbol{\omega}}$.

When Ω is a commutative semigroup (the operation of which is denoted additively), we define the *sum of a non-empty word* as

$$\|\boldsymbol{\omega}\| = \omega_1 + \cdots + \omega_r \in \Omega, \quad \boldsymbol{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet.$$

Then, for any mould $U^\bullet$ such that $U^\emptyset = 0$, the formula

$$\nabla_{U^\bullet} M^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega} = \boldsymbol{\alpha} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\gamma},\, \boldsymbol{\beta} \neq \emptyset} U^{\boldsymbol{\beta}}\, M^{\boldsymbol{\alpha} \cdot \|\boldsymbol{\beta}\| \cdot \boldsymbol{\gamma}} \tag{4.2}$$

defines a mould derivation $\nabla_{U^\bullet}$. The derivation D_φ is nothing but $\nabla_{U^\bullet}$ with $U^{\boldsymbol{\omega}} = \varphi(\omega_1)$ for $\boldsymbol{\omega} = (\omega_1)$ and $U^{\boldsymbol{\omega}} = 0$ for $r(\boldsymbol{\omega}) \neq 1$.

When $\Omega \subset \boldsymbol{A}$, an important example is

$$\nabla M^{\boldsymbol{\omega}} = \|\boldsymbol{\omega}\| M^{\boldsymbol{\omega}}, \tag{4.3}$$

obtained with $\varphi(\eta) \equiv \eta$. On the other hand, every derivation $d\colon \boldsymbol{A} \to \boldsymbol{A}$ obviously induces a mould derivation D, the action of which on any mould $M^\bullet$ is defined by

$$DM^{\boldsymbol{\omega}} = d(M^{\boldsymbol{\omega}}), \quad \boldsymbol{\omega} \in \Omega^\bullet. \tag{4.4}$$

Remark 4.1. With $\Omega = \mathcal{N}$ defined by (3.2) and $\boldsymbol{A} = \mathbb{C}[[x]]$, the mould $\mathcal{V}^\bullet$ determined in Lemma 3.2 is the unique solution of the mould equation

$$(D + \nabla)\mathcal{V}^\bullet = J_a^\bullet \times \mathcal{V}^\bullet, \tag{4.5}$$

such that $\mathcal{V}^\emptyset = 1$ and $\mathcal{V}^{\boldsymbol{\omega}} \in x\mathbb{C}[[x]]$ for $\boldsymbol{\omega} \neq \emptyset$, with D induced by $d = x^2 \frac{\mathrm{d}}{\mathrm{d}x}$ and

$$J_a^{\boldsymbol{\omega}} = \begin{cases} a_{\omega_1} & \text{if } \boldsymbol{\omega} = (\omega_1), \\ 0 & \text{if } r(\boldsymbol{\omega}) \neq 1. \end{cases} \tag{4.6}$$

4.4. When Ω is a commutative semigroup, the *composition of moulds* is defined as follows:

$$C^\bullet = M^\bullet \circ U^\bullet\colon \quad \emptyset \mapsto C^\emptyset = M^\emptyset,$$
$$\boldsymbol{\omega} \neq \emptyset \mapsto C^{\boldsymbol{\omega}} = \sum_{\substack{s \geq 1,\, \boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^s \neq \emptyset \\ \boldsymbol{\omega} = \boldsymbol{\omega}^1 \cdots \boldsymbol{\omega}^s}} M^{(\|\boldsymbol{\omega}^1\|, \ldots, \|\boldsymbol{\omega}^s\|)} U^{\boldsymbol{\omega}^1} \cdots U^{\boldsymbol{\omega}^s},$$

with summation over all possible decompositions of $\boldsymbol{\omega}$ into non-empty words (thus $1 \leq s \leq r(\boldsymbol{\omega})$ and the sum is finite). The map $M^\bullet \mapsto M^\bullet \circ U^\bullet$ is clearly $\boldsymbol{A}$-linear; it is in fact an $\boldsymbol{A}$-algebra homomorphism:

$$(M^\bullet \circ U^\bullet) \times (N^\bullet \circ U^\bullet) = (M^\bullet \times N^\bullet) \circ U^\bullet$$

(the verification of this distributivity property is left as an exercise).

Obviously, $1^\bullet \circ U^\bullet = 1^\bullet$ for any mould $U^\bullet$. The *identity mould*

$$I^\bullet \colon \boldsymbol{\omega} \mapsto I^{\boldsymbol{\omega}} = \begin{cases} 1 & \text{if } r(\boldsymbol{\omega}) = 1, \\ 0 & \text{if } r(\boldsymbol{\omega}) \neq 1 \end{cases}$$

satisfies $M^\bullet \circ I^\bullet = M^\bullet$ for any mould $M^\bullet$. But $I^\bullet \circ U^\bullet = U^\bullet$ only if $U^\emptyset = 0$ (a requirement that we could have imposed when defining mould composition, since the value of $U^\emptyset$ is ignored when computing $M^\bullet \circ U^\bullet$); in general, $I^\bullet \circ U^\bullet = U^\bullet - U^\emptyset\, 1^\bullet$.

Mould composition is associative[6] and not commutative. One can check that a mould $U^\bullet$ admits an inverse for composition (a mould $V^\bullet$ such that $V^\bullet \circ U^\bullet = U^\bullet \circ V^\bullet = I^\bullet$) if and only if $U^{\boldsymbol{\omega}}$ is invertible in $\boldsymbol{A}$ whenever $r(\boldsymbol{\omega}) = 1$ and $U^\emptyset = 0$. These moulds thus form a group under composition.

In the following, we do not always assume Ω to be a commutative semigroup and mould composition is thus not always defined. However, observe that, in the absence of semigroup structure, the definition of $M^\bullet \circ U^\bullet$ makes sense for any mould $M^\bullet$ such that $M^{\boldsymbol{\omega}}$ only depends on $r(\boldsymbol{\omega})$ and that most of the above properties can be adapted to this particular situation.

4.5. As an elementary illustration, one can express the multiplicative inverse of a mould $M^\bullet$ with $\mu = M^\emptyset$ invertible as

$$(M^\bullet)^{\times(-1)} = G^\bullet \circ M^\bullet, \quad \text{with } G^{\boldsymbol{\omega}} = (-1)^{r(\boldsymbol{\omega})} \mu^{-r(\boldsymbol{\omega})-1}.$$

Indeed, $G^\bullet$ is nothing but the multiplicative inverse of $\mu\, 1^\bullet + I^\bullet$ and

$$M^\bullet = \mu\, 1^\bullet + I^\bullet \circ M^\bullet = (\mu\, 1^\bullet + I^\bullet) \circ M^\bullet,$$

whence the result follows immediately.

The above computation does not require any semigroup structure on Ω. Besides, one can also write $(M^\bullet)^{\times(-1)} = \sum_{s \ge 0} (-1)^s \mu^{-s-1} (M^\bullet - \mu\, 1^\bullet)^{\times s}$ (convergent series for the topology of $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ induced by ord).

4.6. We define elementary scalar moulds $\exp_t^\bullet$, $t \in \mathbb{C}$, and $\log^\bullet$ by the formulas $\exp_t^{\boldsymbol{\omega}} = \frac{t^{r(\boldsymbol{\omega})}}{r(\boldsymbol{\omega})!}$ and

$$\log^{\boldsymbol{\omega}} = 0 \quad \text{if } \boldsymbol{\omega} = \emptyset, \quad \log^{\boldsymbol{\omega}} = \frac{(-1)^{r(\boldsymbol{\omega})-1}}{r(\boldsymbol{\omega})} \quad \text{if } \boldsymbol{\omega} \neq \emptyset.$$

[6] Hint: The computation of $M^\bullet \circ (U^\bullet \circ V^\bullet)$ at $\boldsymbol{\omega}$ involves all the decompositions $\boldsymbol{\omega} = \boldsymbol{\omega}^1 \cdots \boldsymbol{\omega}^s$ into non-empty words and then all the decompositions of each factor $\boldsymbol{\omega}^i$ as $\boldsymbol{\omega}^1 = \boldsymbol{\alpha}^1 \cdots \boldsymbol{\alpha}^{i_1}, \boldsymbol{\omega}^2 = \boldsymbol{\alpha}^{i_1+1} \cdots \boldsymbol{\alpha}^{i_2}, \dots, \boldsymbol{\omega}^s = \boldsymbol{\alpha}^{i_{s-1}+1} \cdots \boldsymbol{\alpha}^{i_s}$ (where $1 \le i_1 < i_2 < \cdots < i_s = t$, with each $\boldsymbol{\alpha}^j$ non-empty); it is equivalent to sum first over all the decompositions $\boldsymbol{\omega} = \boldsymbol{\alpha}^1 \cdots \boldsymbol{\alpha}^t$ and then to consider all manners of regrouping adjacent factors $(\boldsymbol{\alpha}^1 \cdots \boldsymbol{\alpha}^{i_1}) \cdot (\boldsymbol{\alpha}^{i_1+1} \cdots \boldsymbol{\alpha}^{i_2}) \cdots (\boldsymbol{\alpha}^{i_{s-1}+1} \cdots \boldsymbol{\alpha}^{i_s})$, which yields the value of $(M^\bullet \circ U^\bullet) \circ V^\bullet$ at $\boldsymbol{\omega}$.

One can check that

$$\exp_0^\bullet = 1^\bullet, \quad \exp_{t_1}^\bullet \times \exp_{t_2}^\bullet = \exp_{t_1+t_2}^\bullet, \quad t_1, t_2 \in \mathbb{C},$$
$$(\exp_t^\bullet - 1^\bullet) \circ \frac{1}{t} \log^\bullet = \frac{1}{t} \log^\bullet \circ (\exp_t^\bullet - 1^\bullet) = I^\bullet, \quad t \in \mathbb{C}^*$$

(use for instance $\exp_t^\bullet = \sum_{s \ge 0} \frac{t^s}{s!} (I^\bullet)^{\times s}$ and $\log^\bullet = \sum_{s \ge 1} \frac{(-1)^{s-1}}{s} (I^\bullet)^{\times s}$; mould composition is well-defined here even if Ω is not a semigroup).

Now, consider on the one hand the Lie algebra

$$\begin{aligned} \mathfrak{L}^\bullet(\Omega, \boldsymbol{A}) = \{\, U^\bullet \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A}) \mid U^\emptyset = 0 \,\}, \\ \text{with bracketing } [U^\bullet, V^\bullet] = U^\bullet \times V^\bullet - V^\bullet \times U^\bullet, \end{aligned} \tag{4.7}$$

and on the other hand the subgroup

$$G^\bullet(\Omega, \boldsymbol{A}) = \{\, M^\bullet \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A}) \mid M^\emptyset = 1 \,\} \tag{4.8}$$

of the multiplicative group of invertible moulds.

Then, for each $U^\bullet \in \mathfrak{L}^\bullet(\Omega, \boldsymbol{A})$, $(\exp_t^\bullet \circ U^\bullet)_{t \in \mathbb{C}}$ is a one-parameter group inside $G^\bullet(\Omega, \boldsymbol{A})$. Moreover, the map

$$E_t \colon U^\bullet \in \mathfrak{L}^\bullet(\Omega, \boldsymbol{A}) \mapsto M^\bullet = \exp_t^\bullet \circ U^\bullet \in G^\bullet(\Omega, \boldsymbol{A})$$

is a bijection for each $t \in \mathbb{C}^*$ (with reciprocal $M^\bullet \mapsto \frac{1}{t} \log^\bullet \circ U^\bullet$), which allows us to consider $\mathfrak{L}^\bullet(\Omega, \boldsymbol{A})$ as the Lie algebra of $G^\bullet(\Omega, \boldsymbol{A})$ in the sense that

$$[U^\bullet, V^\bullet] = \frac{\mathrm{d}}{\mathrm{d}t} \big(E_t(U^\bullet) \times V^\bullet \times E_t(U^\bullet)^{\times(-1)} \big)_{|t = 0}.$$

Observe that mould composition is not necessary to define the map E_t and its reciprocal: one can use the series

$$E_t(U^\bullet) = \sum_{s \ge 0} \frac{t^s}{s!} (U^\bullet)^{\times s}, \quad E_t^{-1}(M^\bullet) = \frac{1}{t} \sum_{s \ge 1} \frac{(-1)^{s-1}}{s} (M^\bullet - 1^\bullet)^{\times s} \tag{4.9}$$

(they are formally convergent because $\operatorname{ord}(U^\bullet)$ and $\operatorname{ord}(M^\bullet - 1^\bullet) \ge 1$).

5 Alternality and symmetrality

5.1. Even if Ω is not a semigroup, another operation available in $\Omega^\bullet$ is *shuffling*: given two non-empty words $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$, one says that a word $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ belongs to their shuffling if one can write $\boldsymbol{\omega}^1 = (\omega_{\sigma(1)}, \dots, \omega_{\sigma(\ell)})$ and $\boldsymbol{\omega}^2 = (\omega_{\sigma(\ell+1)}, \dots, \omega_{\sigma(r)})$ with a permutation σ such that $\sigma(1) < \dots < \sigma(\ell)$ and $\sigma(\ell+1) < \dots < \sigma(r)$ (in other words, $\boldsymbol{\omega}$ can be obtained by interdigitating the letters of $\boldsymbol{\omega}^1$ and those of $\boldsymbol{\omega}^2$ while preserving their internal order in $\boldsymbol{\omega}^1$ or $\boldsymbol{\omega}^2$). We denote by $\operatorname{sh}\binom{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2}{\boldsymbol{\omega}}$ the number of such permutations σ, and we set $\operatorname{sh}\binom{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2}{\boldsymbol{\omega}} = 0$ if $\boldsymbol{\omega}$ does not belong to the shuffling of $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$.

In case one of the words $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \boldsymbol{\omega}$ is empty, we extend the definition by $\operatorname{sh}\binom{\emptyset, \boldsymbol{\omega}}{\boldsymbol{\omega}} = \operatorname{sh}\binom{\boldsymbol{\omega}, \emptyset}{\boldsymbol{\omega}} = 1$, the value being 0 in the other cases.

Definition 5.1. A mould $M^\bullet$ is said to be alternal if $M^\emptyset = 0$ and, for any two non-empty words $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2$,

$$\sum_{\boldsymbol{\omega} \in \Omega^\bullet} \operatorname{sh}\begin{pmatrix} \boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \\ \boldsymbol{\omega} \end{pmatrix} M^{\boldsymbol{\omega}} = 0. \tag{5.1}$$

It is said to be symmetral if $M^\emptyset = 1$ and, for any two non-empty words $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2$,

$$\sum_{\boldsymbol{\omega} \in \Omega^\bullet} \operatorname{sh}\begin{pmatrix} \boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \\ \boldsymbol{\omega} \end{pmatrix} M^{\boldsymbol{\omega}} = M^{\boldsymbol{\omega}^1} M^{\boldsymbol{\omega}^2}. \tag{5.2}$$

Of course the above sums always have finite support. For instance, if $\boldsymbol{\omega}^1 = (\omega_1)$ and $\boldsymbol{\omega}^2 = (\omega_2, \omega_3)$, the left-hand side in both previous formulas is $M^{\omega_1,\omega_2,\omega_3} + M^{\omega_2,\omega_1,\omega_3} + M^{\omega_2,\omega_3,\omega_1}$.

The motivation for this definition lies in formula (7.2) below. We shall see in Section 7 the interpretation of alternality or symmetrality in terms of the operators obtained by mould-comould expansions: alternal moulds will be related to the Lie algebra of derivations, symmetral moulds to the group of automorphisms.

Alternal (resp. symmetral) moulds have to do with primitive (resp. group-like) elements of a certain graded cocommutative Hopf algebra, at least when $\boldsymbol{A}$ is a field – see the remark on Lemma 5.3 below.

An obvious example of alternal mould is $I^\bullet$, or any mould $J^\bullet$ such that $J^{\boldsymbol{\omega}} = 0$ for $r(\boldsymbol{\omega}) \neq 1$ (as is the case of $J_a^\bullet$ defined by (4.6)). An elementary example of symmetral mould is $\exp_t^\bullet$ for any $t \in \mathbb{C}$; a non-trivial example is the mould $\mathcal{V}^\bullet$ determined by Lemma 3.2, the symmetrality of which is the object of Proposition 5.5 below. The mould $\log^\bullet$ is not alternal (nor symmetral), but "alternel"; alternelity and symmetrelity are two other types of symmetry introduced by Écalle, parallel to alternality and symmetrality, but we shall not be concerned with them in this text (see however the end of Section 7).

The next paragraphs contain the proof of the following properties:

Proposition 5.1. *Alternal moulds form a Lie subalgebra $\mathfrak{L}^\bullet_{\mathrm{alt}}(\Omega, \boldsymbol{A})$ of the Lie algebra $\mathfrak{L}^\bullet(\Omega, \boldsymbol{A})$ defined by (4.7). Symmetral moulds form a subgroup $G^\bullet_{\mathrm{sym}}(\Omega, \boldsymbol{A})$ of the multiplicative group $G^\bullet(\Omega, \boldsymbol{A})$ defined by (4.8). The map E_t defined by (4.9) induces a bijection from $\mathfrak{L}^\bullet_{\mathrm{alt}}(\Omega, \boldsymbol{A})$ to $G^\bullet_{\mathrm{sym}}(\Omega, \boldsymbol{A})$ for each $t \in \mathbb{C}^*$.*

Proposition 5.2. *Given a mould $M^\bullet$, we define a mould $SM^\bullet = \widetilde{M}^\bullet$ by the formulas*

$$\widetilde{M}^\emptyset = M^\emptyset, \quad \widetilde{M}^{\omega_1,\ldots,\omega_r} = (-1)^r M^{\omega_r,\ldots,\omega_1}, \quad r \geq 1, \ \omega_1, \ldots, \omega_r \in \Omega. \tag{5.3}$$

Then S is an involution and an antihomomorphism of the $\boldsymbol{A}$-algebra $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$, and

$$M^\bullet \text{ alternal } \implies SM^\bullet = -M^\bullet,$$
$$M^\bullet \text{ symmetral } \implies SM^\bullet = (M^\bullet)^{\times(-1)} \text{ (multiplicative inverse)}.$$

Proposition 5.3. *If Ω is a commutative semigroup and $U^\bullet$ is alternal, then*

$$M^\bullet \text{ alternal } \implies M^\bullet \circ U^\bullet \text{ alternal,} \tag{5.4}$$
$$M^\bullet \text{ symmetral } \implies M^\bullet \circ U^\bullet \text{ symmetral.} \tag{5.5}$$

If moreover $U^\bullet$ admits an inverse for composition (i.e. if $U^{\boldsymbol{\omega}}$ has a multiplicative inverse in $\boldsymbol{A}$ whenever $r(\boldsymbol{\omega}) = 1$), then this inverse is alternal itself; thus alternal invertible moulds form a subgroup of the group (for composition) of invertible moulds.

Proposition 5.4. *If D is a mould derivation induced by a derivation of $\boldsymbol{A}$, or of the form D_φ with $\varphi\colon \Omega \to \boldsymbol{A}$, or of the form $\nabla_{J^\bullet}$ with $J^\bullet$ alternal (with the assumption that Ω is a commutative semigroup in this last case), and if $M^\bullet$ is symmetral, then $(DM^\bullet) \times (M^\bullet)^{\times(-1)}$ and $(M^\bullet)^{\times(-1)} \times (DM^\bullet)$ are alternal.*

5.2. The following definition will facilitate the proof of most of these properties and enlighten the connection with derivations and algebra automorphisms to be discussed in Section 7.

Definition 5.2. We call dimould[7] any map $\boldsymbol{M}^{\bullet,\bullet}$ from $\Omega^\bullet \times \Omega^\bullet$ to $\boldsymbol{A}$; its value on $(\boldsymbol{\omega}, \boldsymbol{\eta})$ is denoted $\boldsymbol{M}^{\boldsymbol{\omega},\boldsymbol{\eta}}$. The set of dimoulds is denoted $\mathcal{M}^{\bullet\bullet}(\Omega, \boldsymbol{A})$; when viewed as the large algebra of the monoid $\Omega^\bullet \times \Omega^\bullet$, it is a non-commutative $\boldsymbol{A}$-algebra.

Observe that, the monoid law on $\Omega^\bullet \times \Omega^\bullet$ being

$$\boldsymbol{\varpi}^1 = (\boldsymbol{\omega}^1, \boldsymbol{\eta}^1),\ \boldsymbol{\varpi}^2 = (\boldsymbol{\omega}^2, \boldsymbol{\eta}^2) \implies \boldsymbol{\varpi}^1 \raisebox{0.2ex}{\scriptsize$\bullet$} \boldsymbol{\varpi}^2 = (\boldsymbol{\omega}^1 \raisebox{0.2ex}{\scriptsize$\bullet$} \boldsymbol{\omega}^2, \boldsymbol{\eta}^1 \raisebox{0.2ex}{\scriptsize$\bullet$} \boldsymbol{\eta}^2),$$

the finiteness of the number of decompositions of any $\boldsymbol{\varpi} \in \Omega^\bullet \times \Omega^\bullet$ as $\boldsymbol{\varpi} = \boldsymbol{\varpi}^1 \raisebox{0.2ex}{\scriptsize$\bullet$} \boldsymbol{\varpi}^2$ allows us to consider this large algebra, in which the multiplication is defined by a formula similar to (4.1). The unit of dimould multiplication is $1^{\bullet,\bullet}\colon (\boldsymbol{\omega}, \boldsymbol{\eta}) \mapsto 1$ if $\boldsymbol{\omega} = \boldsymbol{\eta} = \emptyset$ and 0 otherwise.

Lemma 5.1. *The map $\tau\colon M^\bullet \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A}) \mapsto \boldsymbol{M}^{\bullet,\bullet} \in \mathcal{M}^{\bullet\bullet}(\Omega, \boldsymbol{A})$ defined by*

$$\boldsymbol{M}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\boldsymbol{\omega} \in \Omega^\bullet} \operatorname{sh}\binom{\boldsymbol{\alpha},\ \boldsymbol{\beta}}{\boldsymbol{\omega}} M^{\boldsymbol{\omega}}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega^\bullet$$

is an $\boldsymbol{A}$-algebra homomorphism.

[7] Not to be confused with the *bimoulds* introduced by Écalle in connection with multizeta values, which correspond to the case where the set Ω itself is the cartesian product of two sets – see the end of Section 13.

Proof. The map τ is clearly $\boldsymbol{A}$-linear and $\tau(1^\bullet) = 1^{\bullet,\bullet}$. Let $P^\bullet = M^\bullet \times N^\bullet$ and $\boldsymbol{P}^{\bullet,\bullet} = \tau(P^\bullet)$; since

$$\boldsymbol{P}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\boldsymbol{\gamma}^1,\boldsymbol{\gamma}^2\in\Omega^\bullet} \operatorname{sh}\binom{\boldsymbol{\alpha},\ \boldsymbol{\beta}}{\boldsymbol{\gamma}^1\centerdot\boldsymbol{\gamma}^2} M^{\boldsymbol{\gamma}^1} N^{\boldsymbol{\gamma}^2}, \quad \boldsymbol{\alpha},\boldsymbol{\beta}\in\Omega^\bullet,$$

the property $\boldsymbol{P}^{\bullet,\bullet} = \tau(M^\bullet)\times\tau(N^\bullet)$ follows from the identity

$$\operatorname{sh}\binom{\boldsymbol{\alpha},\ \boldsymbol{\beta}}{\boldsymbol{\gamma}^1\centerdot\boldsymbol{\gamma}^2} = \sum_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^1\centerdot\boldsymbol{\alpha}^2,\ \boldsymbol{\beta}=\boldsymbol{\beta}^1\centerdot\boldsymbol{\beta}^2} \operatorname{sh}\binom{\boldsymbol{\alpha}^1,\ \boldsymbol{\beta}^1}{\boldsymbol{\gamma}^1}\operatorname{sh}\binom{\boldsymbol{\alpha}^2,\ \boldsymbol{\beta}^2}{\boldsymbol{\gamma}^2} \tag{5.6}$$

(the verification of which is left to the reader). $\square$

As in the case of moulds, we can define the "order" of a dimould and get a pseudovaluation ord: $\mathcal{M}^{\bullet\bullet}(\Omega,\boldsymbol{A}) \to \mathbb{N}\cup\{\infty\}$: by definition $\operatorname{ord}(\boldsymbol{M}^{\bullet,\bullet}) \geq s$ if $\boldsymbol{M}^{\boldsymbol{\omega},\boldsymbol{\eta}} = 0$ whenever $r(\boldsymbol{\omega}) + r(\boldsymbol{\eta}) < s$. We then get a complete pseudovaluation ring $\big(\mathcal{M}^{\bullet\bullet}(\Omega,\boldsymbol{A}),\operatorname{ord}\big)$ and the homomorphism τ is continuous since $\operatorname{ord}(\tau(M^\bullet)) \geq \operatorname{ord}(M^\bullet)$.

Definition 5.3. We call decomposable a dimould $\boldsymbol{P}^{\bullet,\bullet}$ of the form $\boldsymbol{P}^{\boldsymbol{\omega},\boldsymbol{\eta}} = M^{\boldsymbol{\omega}} N^{\boldsymbol{\eta}}$ (for all $\boldsymbol{\omega},\boldsymbol{\eta}\in\Omega^\bullet$), where $M^\bullet$ and $N^\bullet$ are two moulds. We then use the notation $\boldsymbol{P}^{\bullet,\bullet} = M^\bullet\otimes N^\bullet$.

One can check that the relation

$$(M_1^\bullet\otimes N_1^\bullet)\times(M_2^\bullet\otimes N_2^\bullet) = (M_1^\bullet\times M_2^\bullet)\otimes(N_1^\bullet\times N_2^\bullet) \tag{5.7}$$

holds in $\mathcal{M}^{\bullet\bullet}(\Omega,\boldsymbol{A})$, for any four moulds $M_1^\bullet, N_1^\bullet, M_2^\bullet, N_2^\bullet$.

With this notation for decomposable dimoulds, we can now rephrase Definition 5.1 with the help of the homomorphism τ of Lemma 5.1:

Lemma 5.2. *A mould $M^\bullet$ is alternal iff $\tau(M^\bullet) = M^\bullet\otimes 1^\bullet + 1^\bullet\otimes M^\bullet$. A mould $M^\bullet$ is symmetral iff $M^\emptyset = 1$ and $\tau(M^\bullet) = M^\bullet\otimes M^\bullet$.*

Notice that the image of τ is contained in the set of *symmetric dimoulds*, i.e. those $\boldsymbol{M}^{\bullet,\bullet}$ such that $\boldsymbol{M}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \boldsymbol{M}^{\boldsymbol{\beta},\boldsymbol{\alpha}}$, because of the obvious relation

$$\operatorname{sh}\binom{\boldsymbol{\alpha},\ \boldsymbol{\beta}}{\boldsymbol{\omega}} = \operatorname{sh}\binom{\boldsymbol{\beta},\ \boldsymbol{\alpha}}{\boldsymbol{\omega}}, \quad \boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\omega}\in\Omega^\bullet. \tag{5.8}$$

5.3. *Remark on Definition* 5.3. The tensor product is used here as a mere notation, which is related to the tensor product of $\boldsymbol{A}$-algebras as follows: there is a unique $\boldsymbol{A}$-linear map $\rho\colon \mathcal{M}^\bullet(\Omega,\boldsymbol{A})\otimes_{\boldsymbol{A}}\mathcal{M}^\bullet(\Omega,\boldsymbol{A}) \to \mathcal{M}^{\bullet\bullet}(\Omega,\boldsymbol{A})$ such that $\rho(M^\bullet\otimes N^\bullet)$ is the above dimould $\boldsymbol{P}^{\bullet,\bullet}$. The map ρ is an $\boldsymbol{A}$-algebra homomorphism, according to (5.7), however its injectivity is not obvious when $\boldsymbol{A}$ is not a field, and denoting $\rho(M^\bullet\otimes N^\bullet)$ simply as $M^\bullet\otimes N^\bullet$, as in Definition 5.3, is thus an abuse of notation.

In fact, if $\boldsymbol{A}$ is an integral domain, then the $\boldsymbol{A}$-module $\mathcal{M}^\bullet(\Omega,\boldsymbol{A})$ is torsion-free ($\mu M^\bullet = 0$ implies $\mu = 0$ or $M^\bullet = 0$) and $\operatorname{Ker}\rho$ coincides with the set $\mathcal{T}$ of all torsion elements of $\mathcal{M}^\bullet(\Omega,\boldsymbol{A}) \otimes_{\boldsymbol{A}} \mathcal{M}^\bullet(\Omega,\boldsymbol{A})$. Indeed, for any $\xi \in \mathcal{T}$, there is a non-zero $\mu \in \boldsymbol{A}$ such that $\mu\xi = 0$, thus $\mu\rho(\xi) = 0$ in $\mathcal{M}^{\bullet\bullet}(\Omega,\boldsymbol{A})$, whence $\rho(\xi) = 0$. Conversely, suppose $\xi = \sum_{i=1}^n M_i^\bullet \otimes N_i^\bullet \in \operatorname{Ker}\rho$, where the moulds $M_i^\bullet$ are not all zero; without loss of generality we can suppose $M_n^\bullet \neq 0$ and choose $\boldsymbol{\omega}^1 \in \Omega^\bullet$ such that $\mu_n = M_n^{\boldsymbol{\omega}^1} \neq 0$. Setting $\mu_i = M_i^{\boldsymbol{\omega}^1}$ for the other i's, we get $\sum_{i=1}^n \mu_i N_i^\bullet = 0$, whence $\mu_n\xi = \sum_{i=1}^{n-1}(\mu_n M_i^\bullet - \mu_i M_n^\bullet) \otimes N_i^\bullet$, still with $\mu_n\xi \in \operatorname{Ker}\rho$. By induction on n, one gets a non-zero $\mu \in \boldsymbol{A}$ such that $\mu\xi = 0$.

Therefore, ρ is injective when $\boldsymbol{A}$ is a principal integral domain, as is the case of $\mathbb{C}[[x]]$, because any torsion-free $\boldsymbol{A}$-module is then flat (Bourbaki, *Algèbre commutative*, chap. I, §2, n°4, Prop. 3), hence its tensor product with itself is also torsion-free (by flatness, the injectivity of $\phi\colon M^\bullet \mapsto \mu M^\bullet$, for $\mu \neq 0$, implies the injectivity of $\phi \otimes \mathrm{Id}\colon \xi \mapsto \mu\xi$).

This is a fortiori the case when $\boldsymbol{A}$ is a field; this is used in the remark on Lemma 5.3 below.

5.4. *Proof of Proposition* 5.1. The set $\mathcal{L}^\bullet_{\mathrm{alt}}(\Omega,\boldsymbol{A})$ of alternal moulds is clearly an $\boldsymbol{A}$-submodule of $\mathcal{L}^\bullet(\Omega,\boldsymbol{A})$. Given $U^\bullet$ and $V^\bullet$ in this set, the alternality of $[U^\bullet, V^\bullet]$ is easily checked with the help of Lemma 5.1, formula (5.7) and Lemma 5.2.

Let $M^\bullet$ and $N^\bullet$ be symmetral. The symmetrality of $M^\bullet \times N^\bullet$ follows from Lemma 5.1, formula (5.7) and Lemma 5.2. Similarly, the multiplicative inverse $\widetilde{M}^\bullet$ of $M^\bullet$ satisfies $\tau(\widetilde{M}^\bullet) \times \tau(M^\bullet) = \tau(M^\bullet) \times \tau(\widetilde{M}^\bullet) = \tau(1^\bullet) = 1^{\bullet,\bullet}$, by uniqueness of the multiplicative inverse in $\mathcal{M}^{\bullet\bullet}(\Omega,\boldsymbol{A})$ it follows that $\tau(\widetilde{M}^\bullet) = \widetilde{M}^\bullet \otimes \widetilde{M}^\bullet$ and $\widetilde{M}^\bullet$ is symmetral.

Now let $t \in \mathbb{C}^*$. Suppose first $U^\bullet \in \mathcal{L}^\bullet_{\mathrm{alt}}(\Omega,\boldsymbol{A})$. We check that $M^\bullet = E_t(U^\bullet)$ is symmetral by using the continuity of τ and formula (4.9): $\tau(M^\bullet) = \exp(a+b)$ with $a = tU^\bullet \otimes 1^\bullet$ and $b = 1^\bullet \otimes tU^\bullet$, where the exponential series is well-defined in $\mathcal{M}^{\bullet\bullet}(\Omega,\boldsymbol{A})$ because $\operatorname{ord}(a), \operatorname{ord}(b) > 0$; since $a \times b = b \times a$, the standard properties of the exponential series yield

$$\tau(M^\bullet) = \exp(a) \times \exp(b) = (\exp(tU^\bullet) \otimes 1^\bullet) \times \big(1^\bullet \otimes \exp(tU^\bullet)\big) = M^\bullet \otimes M^\bullet.$$

Conversely, supposing $M^\bullet = 1^\bullet + N^\bullet \in G_{\mathrm{sym}}(\Omega,\boldsymbol{A})$, we check that $U^\bullet = E_t^{-1}(M^\bullet)$ is alternal: by continuity, we can apply τ termwise to the logarithm series in (4.9) and write $\tau(M^\bullet - 1^\bullet) = N^\bullet \otimes 1^\bullet + 1^\bullet \otimes N^\bullet + N^\bullet \otimes N^\bullet = a + b + a \times b$, with $a = N^\bullet \otimes 1^\bullet$ and $b = 1^\bullet \otimes N^\bullet$ commuting in $\mathcal{M}^{\bullet\bullet}(\Omega,\boldsymbol{A})$, the conclusion then follows from the identity

$$\sum_{s\geq 1} \tfrac{(-1)^{s-1}}{s}(a+b+a\times b)^{\times s} = \sum_{s\geq 1} \tfrac{(-1)^{s-1}}{s} a^{\times s} + \sum_{s\geq 1} \tfrac{(-1)^{s-1}}{s} b^{\times s}$$

(which follows from the observation that, given $c \in \mathcal{M}^{\bullet\bullet}(\Omega,\boldsymbol{A})$ with $\operatorname{ord}(c) > 0$, $\sum \tfrac{(-1)^{s-1}}{s} c^{\times s}$ is the only dimould ℓ of positive order such that $\exp(\ell) = 1^{\bullet,\bullet} + c$).

5.5. *Proof of Proposition* 5.2. It is obvious that S is an involution and the identity

$$S(M^\bullet \times N^\bullet) = SN^\bullet \times SM^\bullet, \quad M^\bullet, N^\bullet \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$$

clearly follows from the Definition (4.1) of mould multiplication. Let us define an $\boldsymbol{A}$-linear map

$$\xi : \boldsymbol{M}^{\bullet,\bullet} \in \mathcal{M}^{\bullet\bullet}(\Omega, \boldsymbol{A}) \mapsto P^\bullet = \xi(\boldsymbol{M}^{\bullet,\bullet}) \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$$

by the formula

$$P^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}=\boldsymbol{\alpha}\centerdot\boldsymbol{\beta}} (-1)^{r(\boldsymbol{\alpha})} \boldsymbol{M}^{\widetilde{\boldsymbol{\alpha}},\boldsymbol{\beta}}, \quad \boldsymbol{\omega} \in \Omega^\bullet, \tag{5.9}$$

where $\widetilde{\boldsymbol{\alpha}} = (\omega_i, \dots, \omega_1)$ for $\boldsymbol{\alpha} = (\omega_1, \dots, \omega_i)$ with $i \geq 1$ and $\widetilde{\emptyset} = \emptyset$. Thus

$$P^\emptyset = \boldsymbol{M}^{\emptyset,\emptyset}, \quad P^{(\omega_1)} = \boldsymbol{M}^{\emptyset,(\omega_1)} - \boldsymbol{M}^{(\omega_1),\emptyset},$$
$$P^{(\omega_1,\omega_2)} = \boldsymbol{M}^{\emptyset,(\omega_1,\omega_2)} - \boldsymbol{M}^{(\omega_1),(\omega_2)} + \boldsymbol{M}^{(\omega_2,\omega_1),\emptyset},$$

and so on. The rest of Proposition 5.2 follows from

Lemma 5.3. *For any two moulds $M^\bullet, N^\bullet$, one has*

$$\xi(M^\bullet \otimes N^\bullet) = (SM^\bullet) \times N^\bullet, \tag{5.10}$$
$$\xi \circ \tau(M^\bullet) = M^\emptyset 1^\bullet, \tag{5.11}$$

with the homomorphism τ of Lemma 5.1.

Indeed, if $M^\bullet$ is alternal, then

$$SM^\bullet + M^\bullet = (SM^\bullet) \times 1^\bullet + 1^\bullet \times M^\bullet = \xi(M^\bullet \otimes 1^\bullet + 1^\bullet \otimes M^\bullet) = \xi \circ \tau(M^\bullet) = 0,$$

and if $M^\bullet$ is symmetral, then

$$(SM^\bullet) \times M^\bullet = \xi(M^\bullet \otimes M^\bullet) = \xi \circ \tau(M^\bullet) = 1^\bullet$$

and similarly $M^\bullet \times SM^\bullet = 1^\bullet$ because $SM^\bullet$ is clearly symmetral too.

Proof of Lemma 5.3. Formula (5.10) is obvious. Let $\boldsymbol{M}^{\bullet,\bullet} = \tau(M^\bullet)$ and $P^\bullet = \xi(\boldsymbol{M}^{\bullet,\bullet})$. Clearly $P^\emptyset = \boldsymbol{M}^{\emptyset,\emptyset} = M^\emptyset$. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ with $r \geq 1$: we must show that $P^{\boldsymbol{\omega}} = 0$.

Using the notations $\boldsymbol{\alpha}^i = (\omega_1, \dots, \omega_i)$ and $\boldsymbol{\beta}^i = (\omega_{i+1}, \dots, \omega_r)$ for $0 \leq i \leq r$ (with $\boldsymbol{\alpha}^0 = \boldsymbol{\beta}^r = \emptyset$), we can write $P^{\boldsymbol{\omega}} = \sum_{i=0}^r (-1)^i \boldsymbol{M}^{\widetilde{\boldsymbol{\alpha}}^i,\boldsymbol{\beta}^i}$; we then split the sum

$$\boldsymbol{M}^{\widetilde{\boldsymbol{\alpha}}^i,\boldsymbol{\beta}^i} = \sum_{\boldsymbol{\gamma}} \operatorname{sh}\left(\begin{smallmatrix}\widetilde{\boldsymbol{\alpha}}^i, \boldsymbol{\beta}^i \\ \boldsymbol{\gamma}\end{smallmatrix}\right) M^{\boldsymbol{\gamma}}$$

according to the first letter of the mute variable: $M^{\widetilde{\alpha}^i,\beta^i} = Q_i + R_i$ with

$$Q_i = \sum_{\gamma} \mathrm{sh}\left(\begin{smallmatrix}\widetilde{\alpha}^{i-1},\,\beta^i\\ \gamma\end{smallmatrix}\right) M^{(\omega_i)\bullet\gamma} \quad \text{if } 1 \le i \le r, \qquad Q_0 = 0,$$

$$R_i = \sum_{\gamma} \mathrm{sh}\left(\begin{smallmatrix}\widetilde{\alpha}^{i},\,\beta^{i+1}\\ \gamma\end{smallmatrix}\right) M^{(\omega_{i+1})\bullet\gamma} \quad \text{if } 1 \le i \le r-1, \quad R_r = 0.$$

But, if $0 \le i \le r-1$, $Q_{i+1} = R_i$, whence

$$P^{\omega} = \sum_{i=1}^{r} (-1)^i Q_i + \sum_{i=0}^{r-1} (-1)^i Q_{i+1} = 0.$$

5.6. *Remark on Lemma* 5.3. Although this will not be used in the rest of the article, it is worth noting here that the structure we have on $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ is very reminiscent of that of a *cocommutative Hopf algebra*: the algebra structure is given by mould multiplication (4.1), with its unit $1^\bullet$; as for the cocommutative coalgebra structure, we may think of the map $\varepsilon \colon M^\bullet \mapsto M^\emptyset$ as of a *counit* and of the homomorphism τ as of a kind of *coproduct* (although its range is not exactly $\mathcal{M}^\bullet(\Omega, \boldsymbol{A}) \otimes_{\boldsymbol{A}} \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$); we now may consider that the involution $S \colon M^\bullet \mapsto \widetilde{M}^\bullet$ behaves as an *antipode*.

Indeed, the identity[8] $\tau(M^\bullet)^{\emptyset,\alpha} = \tau(M^\bullet)^{\alpha,\emptyset} = M^{\alpha}$ can be interpreted as a counit-like property for ε and the fact that any dimould in the image of ρ is symmetric (consequence of (5.8)) as a cocommutativity-like property, in the sense that $\tau(M^\bullet) = \sum P_i^\bullet \otimes Q_i^\bullet$ implies $\sum \varepsilon(P_i^\bullet) Q_i^\bullet = \sum \varepsilon(Q_i^\bullet) P_i^\bullet = M^\bullet$ and $\sum P_i^\bullet \otimes Q_i^\bullet = \sum Q_i^\bullet \otimes P_i^\bullet$. The analogue of coassociativity for τ is obtained by considering the maps τ_ℓ and τ_r which associate with any dimould $\boldsymbol{M}^{\bullet,\bullet}$ the "trimoulds" $\boldsymbol{P}^{\bullet,\bullet,\bullet} = \tau_\ell(\boldsymbol{M}^{\bullet,\bullet})$ and $\boldsymbol{Q}^{\bullet,\bullet,\bullet} = \tau_r(\boldsymbol{M}^{\bullet,\bullet})$ defined by

$$\boldsymbol{P}^{\alpha,\beta,\gamma} = \sum_{\eta\in\Omega^\bullet} \mathrm{sh}\left(\begin{smallmatrix}\alpha,\,\beta\\ \eta\end{smallmatrix}\right) \boldsymbol{M}^{\eta,\gamma}, \quad \boldsymbol{Q}^{\alpha,\beta,\gamma} = \sum_{\eta\in\Omega^\bullet} \mathrm{sh}\left(\begin{smallmatrix}\beta,\,\gamma\\ \eta\end{smallmatrix}\right) \boldsymbol{M}^{\alpha,\eta}$$

and by observing[9] that $\tau_\ell \circ \tau = \tau_r \circ \tau$: when $\tau(M^\bullet) = \sum_i P_i^\bullet \otimes Q_i^\bullet$ with $\tau(P_i^\bullet) = \sum_j A_{i,j}^\bullet \otimes B_{i,j}^\bullet$ and $\tau(Q_i^\bullet) = \sum_k C_{i,k}^\bullet \otimes D_{i,k}^\bullet$, this yields

$$\sum_{i,j} A_{i,j}^\bullet \otimes B_{i,j}^\bullet \otimes Q_i^\bullet = \sum_{i,k} P_i^\bullet \otimes C_{i,k}^\bullet \otimes D_{i,k}^\bullet.$$

Finally, the compatibility of ε, τ and S is expressed through formulas (5.10)–(5.11) (complemented by relations $\xi'(M^\bullet \otimes N^\bullet) = M^\bullet \times SN^\bullet$ and $\xi' \circ \tau(M^\bullet) = M^\emptyset 1^\bullet$ involving a map ξ' defined by replacing $(-1)^{r(\alpha)} \boldsymbol{M}^{\widetilde{\alpha},\beta}$ with $(-1)^{r(\beta)} \boldsymbol{M}^{\alpha,\widetilde{\beta}}$ in (5.9));

[8] derived from the relation $\mathrm{sh}\left(\begin{smallmatrix}\emptyset,\,\alpha\\ \omega\end{smallmatrix}\right) = \mathrm{sh}\left(\begin{smallmatrix}\alpha,\,\emptyset\\ \omega\end{smallmatrix}\right) = 1_{\{\omega=\alpha\}}$.

[9] Proof: for a mould $\boldsymbol{M}^\bullet$, we have $\tau_\ell \circ \tau(\boldsymbol{M}^\bullet)^{\alpha,\beta,\gamma} = \sum_{\omega} \mathrm{sh}\left(\begin{smallmatrix}\alpha,\,\beta,\,\gamma\\ \omega\end{smallmatrix}\right) \boldsymbol{M}^{\omega}$ with $\mathrm{sh}\left(\begin{smallmatrix}\alpha,\,\beta,\,\gamma\\ \omega\end{smallmatrix}\right) = \sum_{\eta} \mathrm{sh}\left(\begin{smallmatrix}\alpha,\,\beta\\ \eta\end{smallmatrix}\right) \mathrm{sh}\left(\begin{smallmatrix}\eta,\,\gamma\\ \omega\end{smallmatrix}\right)$ coinciding with $\sum_{\eta} \mathrm{sh}\left(\begin{smallmatrix}\alpha,\,\eta\\ \omega\end{smallmatrix}\right) \mathrm{sh}\left(\begin{smallmatrix}\beta,\,\gamma\\ \eta\end{smallmatrix}\right)$, and therefore $\tau_r \circ \tau(\boldsymbol{M}^\bullet)^{\alpha,\beta,\gamma} = \sum_{\omega} \mathrm{sh}\left(\begin{smallmatrix}\alpha,\,\beta,\,\gamma\\ \omega\end{smallmatrix}\right) \boldsymbol{M}^{\omega}$ as well.

therefore

$$\tau(M^\bullet) = \sum_i P_i^\bullet \otimes Q_i^\bullet \implies \sum SP_i^\bullet \times Q_i^\bullet = M^\emptyset 1^\bullet = \sum P_i^\bullet \times SQ_i^\bullet.$$

When $\boldsymbol{A}$ is a field, we get a true cocommutative Hopf algebra (graded by ord) by considering $\mathcal{H}^\bullet(\Omega, \boldsymbol{A}) = \tau^{-1}(\mathcal{B})$ with $\mathcal{B} = \mathcal{M}^\bullet(\Omega, \boldsymbol{A}) \otimes_{\boldsymbol{A}} \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ (we can view $\mathcal{B}$ as a subalgebra of $\mathcal{M}^{\bullet\bullet}(\Omega, \boldsymbol{A})$ according to the remark on Definition 5.3). Indeed, in view of the above, it suffices essentially to check that $M^\bullet \in \mathcal{H} = \mathcal{H}^\bullet(\Omega, \boldsymbol{A})$ implies $\tau(M^\bullet) \in \mathcal{H} \otimes_{\boldsymbol{A}} \mathcal{H}$ (and not only $\tau(M^\bullet) \in \mathcal{B}$), so that the restriction of the homomorphism τ to $\mathcal{H}$ is a *bona fide* coproduct

$$\Delta \colon \mathcal{H} \to \mathcal{H} \otimes_{\boldsymbol{A}} \mathcal{H}.$$

This can be done by choosing a minimal N such that $\tau(M^\bullet)$ can be written as a sum of N decomposable dimoulds: $\tau(M^\bullet) = \sum_{i=1}^N P_i^\bullet \otimes Q_i^\bullet$ then implies that the $Q_i^\bullet$'s are linearly independent over $\boldsymbol{A}$ and the coassociativity property allows one to show that each $P_i^\bullet$ lies in $\mathcal{H}$ (choose a basis of $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$, the first N vectors of which are $Q_1^\bullet, \dots, Q_N^\bullet$, and call $\xi_1, \dots, \xi_N$ the first N covectors of the dual basis: the coassociativity identity can be written $\sum_i \tau(P_i^\bullet)^{\boldsymbol{\alpha},\boldsymbol{\beta}} Q_i^{\boldsymbol{\gamma}} = \sum_j P_i^{\boldsymbol{\alpha}} \tau(Q_j^\bullet)^{\boldsymbol{\beta},\boldsymbol{\gamma}}$, thus $\tau(P_i^\bullet) = \sum_j P_j^\bullet \otimes N_{i,j}^\bullet$ with $N_{i,j}^{\boldsymbol{\beta}} = \xi_i\big(\tau(Q_j^\bullet)^{\boldsymbol{\beta},\bullet}\big)$, hence $P_i^\bullet \in \mathcal{H}$); similarly each $Q_i^\bullet$ lies in $\mathcal{H}$.

By definition, all the alternal and symmetral moulds belong to this Hopf algebra $\mathcal{H}$, in which they appear respectively as *primitive* and *group-like* elements.

Finally, when $\boldsymbol{A}$ is only supposed to be an integral domain, $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ can be viewed as a subalgebra of $\mathcal{M}^\bullet(\Omega, K)$, where K denotes the fraction field of $\boldsymbol{A}$; the $\boldsymbol{A}$-valued alternal and symmetral moulds belong to the corresponding Hopf algebra $\mathcal{H}^\bullet(\Omega, K)$.

5.7. *Proof of Proposition* 5.3. The structure of commutative semigroup on Ω allows us to define a composition involving a dimould and a mould as follows: $\boldsymbol{C}^{\bullet,\bullet} = \boldsymbol{M}^{\bullet,\bullet} \circ U^\bullet$ if, for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega^\bullet$,

$$\boldsymbol{C}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum \boldsymbol{M}^{(\|\boldsymbol{\alpha}^1\|,\dots,\|\boldsymbol{\alpha}^s\|),(\|\boldsymbol{\beta}^1\|,\dots,\|\boldsymbol{\beta}^t\|)} U^{\boldsymbol{\alpha}^1} \cdots U^{\boldsymbol{\alpha}^s} U^{\boldsymbol{\beta}^1} \cdots U^{\boldsymbol{\beta}^t},$$

with summation over all possible decompositions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ into non-empty words; when $\boldsymbol{\alpha}$ is the empty word, the convention is to replace $(\|\boldsymbol{\alpha}^1\|, \dots, \|\boldsymbol{\alpha}^s\|)$ by $\emptyset$ and $U^{\boldsymbol{\alpha}^1} \cdots U^{\boldsymbol{\alpha}^s}$ by 1, and similarly when $\boldsymbol{\beta}$ is the empty word.

One can check that $\boldsymbol{M}^{\bullet,\bullet} \circ I^\bullet = \boldsymbol{M}^{\bullet,\bullet}$ and $\boldsymbol{M}^{\bullet,\bullet} \circ (U^\bullet \circ V^\bullet) = (\boldsymbol{M}^{\bullet,\bullet} \circ U^\bullet) \circ V^\bullet$ for any dimould $\boldsymbol{M}^{\bullet,\bullet}$ and any two moulds $U^\bullet, V^\bullet$ (by the same argument as for the associativity of mould composition).

Proposition 5.3 will follow from

Lemma 5.4. *For any three moulds $M^\bullet, N^\bullet, U^\bullet$,*

$$(M^\bullet \otimes N^\bullet) \circ U^\bullet = (M^\bullet \circ U^\bullet) \otimes (N^\bullet \circ U^\bullet). \tag{5.12}$$

For any two moulds $M^\bullet, U^\bullet$,

$$U^\bullet \text{ alternal} \implies \tau(M^\bullet \circ U^\bullet) = \tau(M^\bullet) \circ U^\bullet. \tag{5.13}$$

Proof. The identity (5.12) is an easy consequence of the definition of mould composition in Section 4. As for (5.13), let us suppose $U^\bullet$ alternal and let $\boldsymbol{M}^{\bullet,\bullet} = \tau(M^\bullet)$, $\boldsymbol{U}^{\bullet,\bullet} = \tau(U^\bullet)$, $\boldsymbol{C}^{\bullet,\bullet} = \tau(M^\bullet \circ U^\bullet)$. We have $\boldsymbol{C}^{\emptyset,\emptyset} = M^\emptyset = \boldsymbol{M}^{\emptyset,\emptyset}$, as desired. Suppose now $\boldsymbol{\alpha}$ or $\boldsymbol{\beta} \neq \emptyset$, then

$$\boldsymbol{C}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{s\geq 1,\, \boldsymbol{\gamma}^1,\dots,\boldsymbol{\gamma}^s \neq \emptyset} \operatorname{sh}\begin{pmatrix} \boldsymbol{\alpha},\ \boldsymbol{\beta} \\ \boldsymbol{\gamma}^1 \cdots \boldsymbol{\gamma}^s \end{pmatrix} M^{(\|\boldsymbol{\gamma}^1\|,\dots,\|\boldsymbol{\gamma}^s\|)} U^{\boldsymbol{\gamma}^1} \cdots U^{\boldsymbol{\gamma}^s}.$$

Using the identity (which is an easy generalisation of (5.6))

$$\operatorname{sh}\begin{pmatrix} \boldsymbol{\alpha},\ \boldsymbol{\beta} \\ \boldsymbol{\gamma}^1 \cdots \boldsymbol{\gamma}^s \end{pmatrix} = \sum_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^1\cdots\boldsymbol{\alpha}^s,\, \boldsymbol{\beta}=\boldsymbol{\beta}^1\cdots\boldsymbol{\beta}^s} \operatorname{sh}\begin{pmatrix} \boldsymbol{\alpha}^1,\ \boldsymbol{\beta}^1 \\ \boldsymbol{\gamma}^1 \end{pmatrix} \cdots \operatorname{sh}\begin{pmatrix} \boldsymbol{\alpha}^s,\ \boldsymbol{\beta}^s \\ \boldsymbol{\gamma}^s \end{pmatrix}, \tag{5.14}$$

with possibly empty factors $\boldsymbol{\alpha}^i, \boldsymbol{\beta}^i$, we get

$$\boldsymbol{C}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\substack{s\geq 1,\, \boldsymbol{\alpha}=\boldsymbol{\alpha}^1\cdots\boldsymbol{\alpha}^s, \\ \boldsymbol{\beta}=\boldsymbol{\beta}^1\cdots\boldsymbol{\beta}^s}} M^{(\|\boldsymbol{\alpha}^1\|+\|\boldsymbol{\beta}^1\|,\dots,\|\boldsymbol{\alpha}^s\|+\|\boldsymbol{\beta}^s\|)} \boldsymbol{U}^{\boldsymbol{\alpha}^1,\boldsymbol{\beta}^1} \cdots \boldsymbol{U}^{\boldsymbol{\alpha}^s,\boldsymbol{\beta}^s}, \tag{5.15}$$

with the convention $\|\emptyset\| = 0$. Observe that this last summation involves only finitely many nonzero terms because $\boldsymbol{\alpha}^i = \boldsymbol{\beta}^i = \emptyset$ implies $\boldsymbol{U}^{\boldsymbol{\alpha}^i,\boldsymbol{\beta}^i} = 0$.

If $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ is the empty word, since $\boldsymbol{U}^{\boldsymbol{\omega},\emptyset} = \boldsymbol{U}^{\emptyset,\boldsymbol{\omega}} = U^{\boldsymbol{\omega}}$ we obtain that the values of $\boldsymbol{C}^{\bullet,\bullet}$ and $\boldsymbol{M}^{\bullet,\bullet} \circ U^\bullet$ at $(\boldsymbol{\alpha},\boldsymbol{\beta})$ coincide. If neither $\boldsymbol{\alpha}$ nor $\boldsymbol{\beta}$ is empty, then we have moreover $\boldsymbol{U}^{\boldsymbol{\alpha}^i,\boldsymbol{\beta}^i} \neq 0 \implies \boldsymbol{\alpha}^i$ or $\boldsymbol{\beta}^i = \emptyset$, thus (5.15) can be rewritten (retaining only non-empty factors)

$$\boldsymbol{C}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum \sum_{\boldsymbol{\omega}\in\Omega^\bullet} \operatorname{sh}\left({}^{(\|\boldsymbol{\alpha}^1\|,\dots,\|\boldsymbol{\alpha}^s\|),\, (\|\boldsymbol{\beta}^1\|,\dots,\|\boldsymbol{\beta}^t\|)}_{\boldsymbol{\omega}} \right) M^{\boldsymbol{\omega}} U^{\boldsymbol{\alpha}^1} \cdots U^{\boldsymbol{\alpha}^s} U^{\boldsymbol{\beta}^1} \cdots U^{\boldsymbol{\beta}^t},$$

with the first summation over all possible decompositions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ into non-empty words. We thus get the desired result. □

End of the proof of Proposition 5.3. We now suppose that $U^\bullet$ is an alternal mould. If $M^\bullet$ is an alternal mould, then

$$\tau(M^\bullet \circ U^\bullet) = (M^\bullet \otimes 1^\bullet + 1^\bullet \otimes M^\bullet) \circ U^\bullet = (M^\bullet \circ U^\bullet) \otimes 1^\bullet + 1^\bullet \otimes (M^\bullet \circ U^\bullet)$$

by (5.12)–(5.13), while, for $M^\bullet$ symmetral,

$$\tau(M^\bullet \circ U^\bullet) = (M^\bullet \otimes M^\bullet) \circ U^\bullet = (M^\bullet \circ U^\bullet) \otimes (M^\bullet \circ U^\bullet).$$

Finally, if moreover $U^\bullet$ is invertible for composition and $V^\bullet = (U^\bullet)^{\circ(-1)}$, then $\tau(V^\bullet) = \tau(V^\bullet) \circ (U^\bullet \circ V^\bullet) = \big(\tau(V^\bullet) \circ U^\bullet\big) \circ V^\bullet = \tau(V^\bullet \circ U^\bullet) \circ V^\bullet = (I^\bullet \otimes 1^\bullet + 1^\bullet \otimes I^\bullet) \circ V^\bullet = V^\bullet \otimes 1^\bullet + 1^\bullet \otimes V^\bullet$.

5.8. *Proof of Proposition* 5.4. Let $M^\bullet$ be a symmetral mould. If D is induced by a derivation $d: \boldsymbol{A} \to \boldsymbol{A}$, then we can apply d to both sides of equation (5.2) and we get

$$\tau(DM^\bullet) = DM^\bullet \otimes M^\bullet + M^\bullet \otimes DM^\bullet. \tag{5.16}$$

Let us show that the same relation holds when $D = \nabla_{J^\bullet}$ with $J^\bullet$ alternal (this includes the case $D = D_\varphi$). We set $C^\bullet = \nabla_{J^\bullet} M^\bullet$ and denote respectively by $\boldsymbol{J}^{\bullet,\bullet}$, $\boldsymbol{M}^{\bullet,\bullet}$, $\boldsymbol{C}^{\bullet,\bullet}$ the images of $J^\bullet, M^\bullet, C^\bullet$ by the homomorphism τ. We first observe that $C^\emptyset = 0$ and $\boldsymbol{C}^{\emptyset,\emptyset} = 0$. Let $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \in \Omega^\bullet$ with at least one of them non-empty. From the definition of $C^\bullet$, we have

$$\begin{aligned}
\boldsymbol{C}^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2} &= \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\\ \boldsymbol{\beta}\neq\emptyset}} \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\omega}^1,\boldsymbol{\omega}^2\\ \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\gamma}\end{smallmatrix}\right) M^{\boldsymbol{\alpha}\cdot\|\boldsymbol{\beta}\|\cdot\boldsymbol{\gamma}} J^{\boldsymbol{\beta}}\\
&= \sum_{\substack{\boldsymbol{\omega}^1=\boldsymbol{\alpha}^1\cdot\boldsymbol{\beta}^1\cdot\boldsymbol{\gamma}^1\\ \boldsymbol{\omega}^2=\boldsymbol{\alpha}^2\cdot\boldsymbol{\beta}^2\cdot\boldsymbol{\gamma}^2}} \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\gamma}\\ \boldsymbol{\beta}\neq\emptyset}} \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2\\ \boldsymbol{\alpha}\end{smallmatrix}\right) \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\gamma}^1,\boldsymbol{\gamma}^2\\ \boldsymbol{\gamma}\end{smallmatrix}\right) M^{\boldsymbol{\alpha}\cdot(\|\boldsymbol{\beta}^1\|+\|\boldsymbol{\beta}^2\|)\cdot\boldsymbol{\gamma}} \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\beta}^1,\boldsymbol{\beta}^2\\ \boldsymbol{\beta}\end{smallmatrix}\right) J^{\boldsymbol{\beta}}
\end{aligned}$$

by virtue of (5.14) with $s = 3$. The summation over $\boldsymbol{\beta}$ leads to the appearance of the factor $\boldsymbol{J}^{\boldsymbol{\beta}^1,\boldsymbol{\beta}^2}$. By alternality of $J^\bullet$, this factor vanishes if both $\boldsymbol{\beta}^1$ and $\boldsymbol{\beta}^2$ are non-empty, thus

$$\begin{aligned}
\boldsymbol{C}^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2} = &\sum_{\substack{\boldsymbol{\omega}^1=\boldsymbol{\alpha}^1\cdot\boldsymbol{\beta}^1\cdot\boldsymbol{\gamma}^1,\\ \boldsymbol{\beta}^1\neq\emptyset}} \Phi_1(\boldsymbol{\alpha}^1, \|\boldsymbol{\beta}^1\|, \boldsymbol{\gamma}^1; \boldsymbol{\omega}^2) J^{\boldsymbol{\beta}^1}\\
&+ \sum_{\substack{\boldsymbol{\omega}^2=\boldsymbol{\alpha}^2\cdot\boldsymbol{\beta}^2\cdot\boldsymbol{\gamma}^2,\\ \boldsymbol{\beta}^2\neq\emptyset}} \Phi_2(\boldsymbol{\omega}^1; \boldsymbol{\alpha}^2, \|\boldsymbol{\beta}^2\|, \boldsymbol{\gamma}^2) J^{\boldsymbol{\beta}^2},
\end{aligned}$$

with $\Phi_1(\boldsymbol{\alpha}^1, b, \boldsymbol{\gamma}^1; \boldsymbol{\omega}^2) = \sum_{\boldsymbol{\alpha},\boldsymbol{\gamma},\, \boldsymbol{\omega}^2=\boldsymbol{\alpha}^2\cdot\boldsymbol{\gamma}^2} \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2\\ \boldsymbol{\alpha}\end{smallmatrix}\right) \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\gamma}^1,\boldsymbol{\gamma}^2\\ \boldsymbol{\gamma}\end{smallmatrix}\right) M^{\boldsymbol{\alpha}\cdot b\cdot\boldsymbol{\gamma}}$ and a symmetric definition for Φ_2. A moment of thought shows that

$$\Phi_1(\boldsymbol{\alpha}^1, b, \boldsymbol{\gamma}^1; \boldsymbol{\omega}^2) = \sum_{\boldsymbol{\omega}} \operatorname{sh}\begin{pmatrix}\boldsymbol{\alpha}^1\cdot b\cdot\boldsymbol{\gamma}^1,\ \boldsymbol{\omega}^2\\ \boldsymbol{\omega}\end{pmatrix} M^{\boldsymbol{\omega}} = \boldsymbol{M}^{\boldsymbol{\alpha}^1\cdot b\cdot\boldsymbol{\gamma}^1,\boldsymbol{\omega}^2},$$

with a symmetric formula for Φ_2, so that

$$\boldsymbol{C}^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2} = \sum_{\boldsymbol{\omega}^1=\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\gamma}} \boldsymbol{M}^{\boldsymbol{\alpha}\cdot\|\boldsymbol{\beta}\|\cdot\boldsymbol{\gamma},\boldsymbol{\omega}^2} U^{\boldsymbol{\beta}} + \sum_{\boldsymbol{\omega}^2=\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\gamma}} \boldsymbol{M}^{\boldsymbol{\omega}^1,\boldsymbol{\alpha}\cdot\|\boldsymbol{\beta}\|\cdot\boldsymbol{\gamma}} U^{\boldsymbol{\beta}},$$

whence formula (5.16) follows.

Since the multiplicative inverse $\widetilde{M}^\bullet$ of $M^\bullet$ is known to be symmetral by Proposition 5.1, we can multiply both sides of (5.16) by $\tau(\widetilde{M}^\bullet)$ and use Lemma 5.1 and formula (5.7); this yields the symmetrality of $DM^\bullet \times \widetilde{M}^\bullet$ and $\widetilde{M}^\bullet \times DM^\bullet$.

5.9. There is a kind of converse to Proposition 5.4, which is essential in the application to the saddle-node; we state it in this context only:

Proposition 5.5. *Let* $\Omega = \mathcal{N}$ *as in* (3.2) *and* $\boldsymbol{A} = \mathbb{C}[[x]]$. *Then the mould* $\mathcal{V}^{\bullet}$ *defined by Lemma* 3.2 *is symmetral.*

Proof. We must show that

$$\mathcal{V}^{\boldsymbol{\alpha}}\mathcal{V}^{\boldsymbol{\beta}} = \sum_{\boldsymbol{\gamma}\in\Omega^{\bullet}} \operatorname{sh}\binom{\boldsymbol{\alpha},\ \boldsymbol{\beta}}{\boldsymbol{\gamma}} \mathcal{V}^{\boldsymbol{\gamma}}, \quad \boldsymbol{\alpha},\boldsymbol{\beta} \in \Omega^{\bullet}. \tag{5.17}$$

Since $\mathcal{V}^{\emptyset} = 1$, this is obviously true for $\boldsymbol{\alpha}$ or $\boldsymbol{\beta} = \emptyset$. We now argue by induction on $r = r(\boldsymbol{\alpha}) + r(\boldsymbol{\beta})$. We thus suppose $r \geq 1$ and, without loss of generality, both of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ non-empty. With the notations $d = x^2\frac{\mathrm{d}}{\mathrm{d}x}$, $\|\boldsymbol{\alpha}\| = \alpha_1 + \cdots + \alpha_{r(\boldsymbol{\alpha})}$ and $\|\boldsymbol{\beta}\| = \beta_1 + \cdots + \beta_{r(\boldsymbol{\beta})}$, we compute

$$\begin{aligned} A &:= (d + \|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\|) \sum_{\boldsymbol{\gamma}} \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\alpha},\boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{smallmatrix}\right)\mathcal{V}^{\boldsymbol{\gamma}} \\ &= \sum_{\boldsymbol{\gamma}\neq\emptyset} \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\alpha},\boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{smallmatrix}\right)(d + \|\boldsymbol{\gamma}\|)\mathcal{V}^{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\gamma}\neq\emptyset} \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\alpha},\boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{smallmatrix}\right)a_{\gamma_1}\mathcal{V}^{`\boldsymbol{\gamma}}, \end{aligned}$$

using the notation $`\boldsymbol{\omega} = (\omega_2,\ldots,\omega_s)$ for any non-empty $\boldsymbol{\omega} = (\omega_1,\ldots,\omega_s)$ and the defining equation of $\mathcal{V}^{\bullet}$. Splitting the last summation according to the value of γ_1, we get

$$A = \sum_{\boldsymbol{\delta}} \operatorname{sh}\left(\begin{smallmatrix}`\boldsymbol{\alpha},\boldsymbol{\beta}\\ \boldsymbol{\delta}\end{smallmatrix}\right)a_{\alpha_1}\mathcal{V}^{\boldsymbol{\delta}} + \sum_{\boldsymbol{\delta}} \operatorname{sh}\left(\begin{smallmatrix}\boldsymbol{\alpha},`\boldsymbol{\beta}\\ \boldsymbol{\delta}\end{smallmatrix}\right)a_{\beta_1}\mathcal{V}^{\boldsymbol{\delta}} = a_{\alpha_1}\mathcal{V}^{`\boldsymbol{\alpha}}\cdot\mathcal{V}^{\boldsymbol{\beta}} + \mathcal{V}^{\boldsymbol{\alpha}}\cdot a_{\beta_1}\mathcal{V}^{`\boldsymbol{\beta}}$$

(using the induction hypothesis), hence

$$A = (d + \|\boldsymbol{\alpha}\|)\mathcal{V}^{\boldsymbol{\alpha}}\cdot\mathcal{V}^{\boldsymbol{\beta}} + \mathcal{V}^{\boldsymbol{\alpha}}\cdot(d + \|\boldsymbol{\beta}\|)\mathcal{V}^{\boldsymbol{\beta}} = (d + \|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\|)(\mathcal{V}^{\boldsymbol{\alpha}}\mathcal{V}^{\boldsymbol{\beta}}).$$

We conclude that both sides of (5.17) must coincide, because $d + \|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\|$ is invertible if $\|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\| \neq 0$ and both of them belong to $x\mathbb{C}[[x]]$, thus even if $\|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\| = 0$ the desired conclusion holds. □

6 General mould-comould expansions

6.1. We still assume that we are given a set Ω and a commutative $\mathbb{C}$-algebra $\boldsymbol{A}$. When Ω is the trivial one-element semigroup $\{0\}$, the algebra of $\boldsymbol{A}$-valued moulds on Ω is nothing but the algebra of formal series $\boldsymbol{A}[[\mathrm{T}]]$, with its usual multiplication and composition laws: the monoid of words is then isomorphic to $\mathbb{N}$ via the map r, and one can identify a mould $M^{\bullet}$ with the generating series $\sum_{\boldsymbol{\omega}\in\Omega^{\bullet}} M^{\boldsymbol{\omega}}\mathrm{T}^{r(\boldsymbol{\omega})}$; it is then easy to check that the above definitions of multiplication and composition boil down to the usual ones.

In the case of a general set Ω, the analogue of this is to identify a mould $M^{\bullet}$ with the element $\sum M^{\omega_1,\ldots,\omega_r}\mathrm{T}_{\omega_1}\cdots\mathrm{T}_{\omega_r}$ of the completion of the free associative (non-commutative) algebra generated by the symbols T_η, $\eta\in\Omega$. When replacing the T_η's by elements B_η of an $\boldsymbol{A}$-algebra, one gets what is called a mould-comould expansion; we now define these objects in a context inspired by Section 3.

6.2. Suppose that $(\mathcal{F}, \mathrm{val})$ is a complete pseudovaluation ring, possibly non-commutative, with unit denoted by Id, such that $\mathcal{F}$ is also an $\boldsymbol{A}$-algebra. We thus have a ring homomorphism $\mu \in \boldsymbol{A} \mapsto \mu \,\mathrm{Id} \in \mathcal{F}$, the image of which lies in the center of $\mathcal{F}$.

Definition 6.1. A comould on Ω with values in $\mathcal{F}$ is any map $\boldsymbol{B}_\bullet \colon \boldsymbol{\omega} \in \Omega^\bullet \mapsto \boldsymbol{B}_{\boldsymbol{\omega}} \in \mathcal{F}$ such that $\boldsymbol{B}_\emptyset = \mathrm{Id}$ and

$$\boldsymbol{B}_{\boldsymbol{\omega}^1 \centerdot \boldsymbol{\omega}^2} = \boldsymbol{B}_{\boldsymbol{\omega}^2} \boldsymbol{B}_{\boldsymbol{\omega}^1}, \quad \boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \in \Omega^\bullet. \tag{6.1}$$

Such an object could even be called *multiplicative comould* to emphasize that the map $\boldsymbol{B}_\bullet \colon \Omega^\bullet \to \mathcal{F}$ is required to be a monoid homomorphism from $\Omega^\bullet$ to the multiplicative monoid underlying the opposite ring of $\mathcal{F}$.

Observe that there is a one-to-one correspondence between comoulds and families $(B_\eta)_{\eta \in \Omega}$ of $\mathcal{F}$ indexed by one-letter words: the formulas $\boldsymbol{B}_\emptyset = \mathrm{Id}$ and $\boldsymbol{B}_{\boldsymbol{\omega}} = B_{\omega_r} \cdots B_{\omega_1}$ for $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r) \in \Omega^\bullet$ with $r \geq 1$ define a comould, which we call the *comould generated by* $(B_\eta)_{\eta \in \Omega}$, and all comoulds are obtained this way.

Suppose a comould $\boldsymbol{B}_\bullet$ is given. For any $\boldsymbol{A}$-valued mould $M^\bullet$ on Ω such that the family $(M^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \Omega^\bullet}$ is formally summable in $\mathcal{F}$ (in particular this family has countable support – cf. Definition 3.2), we can consider the mould-comould expansion, also called *contraction of $M^\bullet$ into $\boldsymbol{B}_\bullet$*,

$$\sum M^\bullet \boldsymbol{B}_\bullet = \sum_{\boldsymbol{\omega} \in \Omega^\bullet} M^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}} \in \mathcal{F}.$$

6.3. The example to keep in mind is related to Definition 3.3. Suppose that $(\mathcal{A}, \nu)$ is any complete pseudovaluation ring such that $\mathcal{A}$ is a commutative $\boldsymbol{A}$-algebra, the unit of which is denoted by 1; thus $\boldsymbol{A}$ is identified to a subalgebra of $\mathcal{A}$ (for instance $(\mathcal{A}, \nu) = (\mathbb{C}[[x, y]], \nu_4)$ and $\boldsymbol{A} = \mathbb{C}[[x]]$). Denote by $\mathcal{E}$ the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ consisting of operators having a valuation with respect to ν, so that $(\mathcal{E}, \mathrm{val}_\nu)$ is a complete pseudovaluation ring. Let

$$\mathcal{F}_{\mathcal{A}, \boldsymbol{A}} = \{\, \Theta \in \mathcal{E} \mid \Theta \text{ and } \mu \,\mathrm{Id} \text{ commute for all } \mu \in \boldsymbol{A} \,\} = \mathcal{E} \cap \mathrm{End}_{\boldsymbol{A}}(\mathcal{A}). \tag{6.2}$$

We get an $\boldsymbol{A}$-algebra, which is a closed subset of $\mathcal{E}$ for the topology induced by val_ν, thus $(\mathcal{F}_{\mathcal{A}, \boldsymbol{A}}, \mathrm{val}_\nu)$ is also a complete pseudovaluation ring; these are the $\boldsymbol{A}$-linear operators of $\mathcal{A}$ having a valuation with respect to ν.

In practice, the B_η's which generate a comould are related to the homogeneous components of an operator of $\mathcal{A}$ that one wishes to analyse. In Section 3 for instance, the derivation $X - X_0$ of $\mathcal{A} = \mathbb{C}[[x, y]]$ was decomposed into a sum of multiples of B_n according to (3.3), where each term $a_n(x) B_n$ is homogeneous of degree n in the sense that it sends $y^{n_0}\mathbb{C}[[x]]$ in $y^{n_0+n}\mathbb{C}[[x]]$ for every n_0. Observe that the commutation of the B_η's with the image of $\boldsymbol{A} = \mathbb{C}[[x]]$ in $\mathcal{E}$ reflects the fact that the vector field $X - X_0$ is "fibred" over the variable x; similarly, one can look for a solution Θ of equation (3.1) in $\mathcal{F}_{\mathcal{A},\boldsymbol{A}}$ because the corresponding formal transformation $(x,y)\mapsto\theta(x,y)$ is expected to be fibred likewise – cf. (2.5).

6.4. Returning to the general situation, we now show how, via mould-comould expansions, mould multiplication corresponds to multiplication in $\mathcal{F}$:

Proposition 6.1. *Suppose that $\boldsymbol{B}_\bullet$ is an $\mathcal{F}$-valued comould on Ω and that $M^\bullet$ and $N^\bullet$ are $\boldsymbol{A}$-valued moulds on Ω such that the families $(M^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet}$ and $(N^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet}$ are formally summable. Then the mould $P^\bullet = M^\bullet\times N^\bullet$ gives rise to a formally summable family $(P^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet}$ and*

$$\sum(M^\bullet\times N^\bullet)\boldsymbol{B}_\bullet = \Big(\sum N^\bullet\boldsymbol{B}_\bullet\Big)\Big(\sum M^\bullet\boldsymbol{B}_\bullet\Big).$$

Proof. Let $\delta_*\in\mathbb{Z}$ such that $v_1(\boldsymbol{\omega}) = \operatorname{val}(M^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}})\geq\delta_*$ and $v_2(\boldsymbol{\omega}) = \operatorname{val}(N^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}})\geq\delta_*$ for all $\boldsymbol{\omega}\in\Omega^\bullet$. Then

$$P^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}=\boldsymbol{\omega}^1\raisebox{0.1ex}{\scriptsize$\bullet$}\boldsymbol{\omega}^2} N^{\boldsymbol{\omega}^2}\boldsymbol{B}_{\boldsymbol{\omega}^2}M^{\boldsymbol{\omega}^1}\boldsymbol{B}_{\boldsymbol{\omega}^1}$$

(since $\boldsymbol{A}$ is a commutative algebra and its image in $\mathcal{F}$ commutes with the $\boldsymbol{B}_{\boldsymbol{\omega}^2}$'s), thus $\operatorname{val}(P^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}})\geq\min\{\,v_1(\boldsymbol{\omega}^1)+v_2(\boldsymbol{\omega}^2)\mid\boldsymbol{\omega}=\boldsymbol{\omega}^1\raisebox{0.1ex}{\scriptsize$\bullet$}\boldsymbol{\omega}^2\,\}\geq 2\delta_*$ and, for any $\delta\in\mathbb{Z}$, the condition $\operatorname{val}(P^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}})\leq\delta$ implies that $\boldsymbol{\omega}$ can be written as $\boldsymbol{\omega}^1\raisebox{0.1ex}{\scriptsize$\bullet$}\boldsymbol{\omega}^2$ with $v_1(\boldsymbol{\omega}^1)\leq\delta-\delta_*$ and $v_2(\boldsymbol{\omega}^2)\leq\delta-\delta_*$, hence they are only finitely many such $\boldsymbol{\omega}$'s.

To compute $\sum P^\bullet\boldsymbol{B}_\bullet$, we can suppose Ω countable (replacing it, if necessary, by the set of all letters appearing in the union of the supports of $(M^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}})$ and $(N^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}})$, which is countable), choose an exhaustion of Ω by finite sets Ω_K, $K\geq 0$, and use $\Omega^{K,R} = \{\,\boldsymbol{\omega}\in\Omega^\bullet\mid r=r(\boldsymbol{\omega})\leq R,\ \omega_1,\dots,\omega_r\in\Omega_K\,\}$, $K,R\geq 0$, as an exhaustion of $\Omega^\bullet$. The conclusion follows from the identity

$$\Big(\sum_{\boldsymbol{\omega}\in\Omega^{K,R}} N^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}}\Big)\Big(\sum_{\boldsymbol{\omega}\in\Omega^{K,R}} M^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}}\Big) - \sum_{\boldsymbol{\omega}\in\Omega^{K,R}} P^{\boldsymbol{\omega}}\boldsymbol{B}_{\boldsymbol{\omega}}$$
$$= \sum_{\substack{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2\in\Omega^{K,R}\\ r(\boldsymbol{\omega}^1)+r(\boldsymbol{\omega}^2)>R}} N^{\boldsymbol{\omega}^2}\boldsymbol{B}_{\boldsymbol{\omega}^2}M^{\boldsymbol{\omega}^1}\boldsymbol{B}_{\boldsymbol{\omega}^1},$$

where the right-hand side tends to 0 as $K,R\to\infty$, since its valuation is at least $\min\{\,v_1(\boldsymbol{\omega}^1)+v_2(\boldsymbol{\omega}^2)\mid\boldsymbol{\omega}^1,\boldsymbol{\omega}^2\in\Omega^{K,R},\ r(\boldsymbol{\omega}^1)+r(\boldsymbol{\omega}^2)>R\,\}\geq\nu_*(K,R)+\delta_*$, with

$$\nu_*(K,R) = \min\{\,\min(v_1(\boldsymbol{\omega}),v_2(\boldsymbol{\omega}))\mid\boldsymbol{\omega}\in\Omega^{K,R},\ r(\boldsymbol{\omega})>R/2\,\}\xrightarrow[R\to\infty]{}\infty$$

for any K (because, for any finite subset F of $\Omega^\bullet$, $\boldsymbol{\omega}\notin F$ as soon as $r(\boldsymbol{\omega})$ is large enough). $\square$

6.5. Suppose Ω is a commutative semigroup. A motivation for the definition of mould composition in Section 4 is

Proposition 6.2. *Suppose that $U^\bullet$ and $M^\bullet$ are moulds such that the families $(U^{\boldsymbol\omega}\boldsymbol{B}_{\boldsymbol\omega})_{\boldsymbol\omega\in\Omega^\bullet}$ and*

$$\Theta_{\boldsymbol\omega^1,\dots,\boldsymbol\omega^s} = M^{\|\boldsymbol\omega^1\|,\dots,\|\boldsymbol\omega^s\|}U^{\boldsymbol\omega^1}\cdots U^{\boldsymbol\omega^s}\boldsymbol{B}_{\boldsymbol\omega^1\cdots\boldsymbol\omega^s}, \quad s\ge 1,\ \boldsymbol\omega^1,\dots,\boldsymbol\omega^s\in\Omega^\bullet$$

are formally summable.[10] *Suppose moreover $U^{\emptyset}=0$, let*

$$B'_\eta = \sum_{\boldsymbol\omega\in\Omega^\bullet \text{ s.t. } \|\boldsymbol\omega\|=\eta} U^{\boldsymbol\omega}\boldsymbol{B}_{\boldsymbol\omega}, \quad \eta\in\Omega, \tag{6.3}$$

and consider the comould $\boldsymbol{B}'_\bullet$ generated by $\{B'_\eta,\ \eta\in\Omega\}$. Then the mould $C^\bullet = M^\bullet\circ U^\bullet$ gives rise to a formally summable family $(C^{\boldsymbol\omega}\boldsymbol{B}_{\boldsymbol\omega})_{\boldsymbol\omega\in\Omega^\bullet}$ and

$$\sum (M^\bullet\circ U^\bullet)\boldsymbol{B}_\bullet = \sum M^\bullet\boldsymbol{B}'_\bullet.$$

Proof. We have $C^{\emptyset}\boldsymbol{B}_\emptyset = M^{\emptyset}\boldsymbol{B}'_\emptyset$, since $C^\emptyset = M^\emptyset$. If $\boldsymbol\omega$ and $\boldsymbol\eta$ are non-empty words in $\Omega^\bullet$, with $\boldsymbol\eta = (\eta_1,\dots,\eta_\sigma)$,

$$C^{\boldsymbol\omega}\boldsymbol{B}_{\boldsymbol\omega} = \sum_{\substack{s\ge1,\,\boldsymbol\omega^1,\dots,\boldsymbol\omega^s\neq\emptyset\\ \boldsymbol\omega=\boldsymbol\omega^1\cdots\boldsymbol\omega^s}} \Theta_{\boldsymbol\omega^1,\dots,\boldsymbol\omega^s}, \qquad M^{\boldsymbol\eta}\boldsymbol{B}'_{\boldsymbol\eta} = \sum_{\substack{\boldsymbol\omega^1,\dots,\boldsymbol\omega^\sigma\neq\emptyset\\ \|\boldsymbol\omega^1\|=\eta_1,\dots,\|\boldsymbol\omega^\sigma\|=\eta_\sigma}} \Theta_{\boldsymbol\omega^1,\dots,\boldsymbol\omega^\sigma}.$$

The conclusion follows easily. □

The idea is that, when indexation by $\eta\in\Omega$ corresponds to a decomposition of an element of $\mathcal{F}$ into homogeneous components, we use the mould $U^\bullet$ to go from $X=\sum_{\eta\in\Omega}B_\eta$ to $Y=\sum_{\eta\in\Omega}B'_\eta$ by contracting it into the comould $\boldsymbol{B}_\bullet$ associated with X; then we use $M^\bullet$ to go from Y to the contraction Z of $M^\bullet$ into the comould $\boldsymbol{B}'_\bullet$ associated with Y. Mould composition thus reflects the composition of these operations on elements of $\mathcal{F}$, $X\mapsto Y$ and $Y\mapsto Z$.

6.6. For example, suppose that $(\boldsymbol{B}_{\boldsymbol\omega})_{\boldsymbol\omega\in\Omega^\bullet}$ is formally summable. Then, in particular, $(B_\eta)_{\eta\in\Omega}$ is formally summable, and $X=\sum B_\eta$ is "exponentiable": for any $t\in\mathbb{C}$, the series $\exp(tX)=\sum_{s\ge0}\frac{t^s}{s!}X^s$ is convergent; moreover, $\exp(tX)=\sum \exp_t^\bullet\boldsymbol{B}_\bullet$. On the other hand, $\mathrm{Id}+X$ has an "infinitesimal generator": the series $Y=\sum_{s\ge1}\frac{(-1)^{s-1}}{s}X^s$ is convergent and $\exp(Y)=\mathrm{Id}+X$; one has $Y=\sum\log^\bullet\boldsymbol{B}_\bullet$.

Now, if $U^\bullet\in\mathfrak{L}^\bullet(\Omega,\boldsymbol{A})$ is such that $(U^{\boldsymbol\omega^1}\cdots U^{\boldsymbol\omega^s}\boldsymbol{B}_{\boldsymbol\omega^1\cdots\boldsymbol\omega^s})_{s\ge1,\boldsymbol\omega^1,\dots,\boldsymbol\omega^s\in\Omega^\bullet}$ is formally summable, then in particular $(U^{\boldsymbol\omega}\boldsymbol{B}_{\boldsymbol\omega})_{\boldsymbol\omega\in\Omega^\bullet}$ is formally summable and $X'=\sum U^\bullet\boldsymbol{B}_\bullet$ is exponentiable, with $\exp(tX')=\sum(\exp_t^\bullet\circ U^\bullet)\boldsymbol{B}_\bullet$ for any $t\in\mathbb{C}$.

Similarly, if $M^\bullet = 1^\bullet + V^\bullet\in G^\bullet(\Omega,\boldsymbol{A})$ with $(V^{\boldsymbol\omega^1}\cdots V^{\boldsymbol\omega^s}\boldsymbol{B}_{\boldsymbol\omega^1\cdots\boldsymbol\omega^s})$ formally summable, then $\sum M^\bullet\boldsymbol{B}_\bullet$ has infinitesimal generator $\sum(\log^\bullet\circ V^\bullet)\boldsymbol{B}_\bullet$.

6.7. For the interpretation of the mould derivations $\nabla_{U^\bullet}$ defined by (4.2), consider a situation similar to that of Proposition 6.2, with a comould $\boldsymbol{B}_\bullet\colon\Omega^\bullet\to\mathcal{F}$, a mould

[10] Notice that the formal summability of the second family follows from the formal summability of the first one when the valuation val on $\mathcal{F}$ only takes non-negative values.

$U^\bullet \in \mathfrak{L}^\bullet(\Omega, \boldsymbol{A})$ such that $(U^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet}$ is formally summable and, for each $\eta \in \Omega$, $B'_\eta \in \mathcal{F}$ still defined by (6.3). But instead of considering the comould $\boldsymbol{B}'_\bullet$ generated by (B'_η), i.e. $\boldsymbol{B}'_{\boldsymbol{\omega}} = B'_{\omega_r} \cdots B'_{\omega_1}$ for $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$, set

$$\boldsymbol{B}'_{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}=\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma},\, r(\boldsymbol{\beta})=1} \boldsymbol{B}_{\boldsymbol{\gamma}} B'_{\boldsymbol{\beta}} \boldsymbol{B}_{\boldsymbol{\alpha}}, \quad \boldsymbol{\omega} \in \Omega^\bullet$$

i.e. $\boldsymbol{B}'_\emptyset = 0$ and $\boldsymbol{B}'_{\boldsymbol{\omega}} = \sum_{i=1}^r B_{\omega_r} \cdots B_{\omega_{i+1}} B'_{\omega_i} B_{\omega_{i-1}} \cdots B_{\omega_1}$ for $r \ge 1$ (beware that $\boldsymbol{B}'_\bullet \colon \Omega^\bullet \to \mathcal{F}$ is not a comould, since multiplicativity fails).

Then one can check the formal summability of $\big((\nabla_{U^\bullet} M^{\boldsymbol{\omega}})\boldsymbol{B}_{\boldsymbol{\omega}}\big)_{\boldsymbol{\omega}\in\Omega^\bullet}$ for any mould $M^\bullet$ such that the families $(M^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet}$ and $(M^{\boldsymbol{\alpha},\|\boldsymbol{\beta}\|,\boldsymbol{\gamma}} U^{\boldsymbol{\beta}} \boldsymbol{B}_{\boldsymbol{\alpha}\boldsymbol{\cdot}\boldsymbol{\beta}\boldsymbol{\cdot}\boldsymbol{\gamma}})_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in\Omega^\bullet}$ are formally summable, with

$$\sum (\nabla_{U^\bullet} M^\bullet)\boldsymbol{B}_\bullet = \sum M^\bullet \boldsymbol{B}'_\bullet.$$

If, moreover, there is an $\boldsymbol{A}$-linear derivation $\mathcal{D} \colon \mathcal{F} \to \mathcal{F}$ such that $\mathcal{D} B_\eta = B'_\eta$ for each $\eta \in \Omega$, then $\boldsymbol{B}'_{\boldsymbol{\omega}}$ is nothing but $\mathcal{D}\boldsymbol{B}_{\boldsymbol{\omega}}$ and the previous identity takes the form

$$\sum (\nabla_{U^\bullet} M^\bullet)\boldsymbol{B}_\bullet = \mathcal{D}\Big(\sum M^\bullet \boldsymbol{B}_\bullet\Big).$$

6.8. For a given commutative algebra $\boldsymbol{A}$, we now consider the case where

$$\Omega \subset \mathbb{Z}^n, \quad \mathcal{A} = \boldsymbol{A}[[y_1, \dots, y_n]],$$

for a fixed $n \in \mathbb{N}^*$.

Definition 6.2. Given $\eta \in \mathbb{Z}^n$ and $\Theta \in \operatorname{End}_{\boldsymbol{A}}(\boldsymbol{A}[[y_1, \dots, y_n]])$, we say that Θ is homogeneous of degree η if $\Theta y^m \in \boldsymbol{A}\, y^{m+\eta}$ for every $m \in \mathbb{N}^n$ (with the usual notation $y^m = y_1^{m_1} \cdots y_n^{m_n}$ for monomials).

For example, any $\lambda \in \boldsymbol{A}^n$ gives rise to an operator

$$\mathcal{X}_\lambda = \lambda_1 y_1 \frac{\partial}{\partial y_1} + \cdots + \lambda_n y_n \frac{\partial}{\partial y_n} \tag{6.4}$$

which is homogeneous of degree 0, since $\mathcal{X}_\lambda y^m = \langle m, \lambda \rangle y^m$.

Suppose moreover that we are given a pseudovaluation $\operatorname{val} \colon \mathcal{A} \to \mathbb{Z} \cup \{\infty\}$ such that $(\mathcal{A}, \operatorname{val})$ is complete and $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ are continuous, and consider $\mathcal{F} = \mathcal{F}_{\mathcal{A},\boldsymbol{A}}$ as defined by (6.2). We suppose $\Omega \subset \mathbb{Z}^n$ because we are interested in $\mathcal{F}$-valued *homogeneous* comoulds, i.e. $\mathcal{F}$-valued comoulds $\boldsymbol{B}_\bullet$ such that $\boldsymbol{B}_{\boldsymbol{\omega}}$ is homogeneous of degree $\|\boldsymbol{\omega}\| = \omega_1 + \cdots + \omega_r \in \mathbb{Z}^n$ for every non-empty $\boldsymbol{\omega} \in \Omega^\bullet$, and homogeneous of degree 0 for $\boldsymbol{\omega} = \emptyset$; in fact, the multiplicativity property (6.1) will not be used for what follows, the following proposition holds for any map $\boldsymbol{B}_\bullet \colon \Omega^\bullet \to \mathcal{F}$ provided it is homogeneous as just defined.

In the case of a comould satisfying the multiplicativity property as required in Definition 6.1, homogeneity is equivalent to the fact that each $B_\eta = \boldsymbol{B}_{(\eta)}$, $\eta \in \Omega$, is homogeneous of degree η.

Proposition 6.3. *Let $\lambda \in \boldsymbol{A}^n$ and $\boldsymbol{B}_\bullet$ be an $\mathcal{F}$-valued homogeneous comould. Then, for every $\boldsymbol{A}$-valued mould $M^\bullet$ such that $(M^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \Omega^\bullet}$ is formally summable,*

$$\Big[\mathcal{X}_\lambda, \sum M^\bullet \boldsymbol{B}_\bullet\Big] = \sum (D_\varphi M^\bullet) \boldsymbol{B}_\bullet, \quad \textit{with } \varphi = \langle\,\cdot\,, \lambda\rangle \colon \Omega \to \boldsymbol{A}. \tag{6.5}$$

Thus, this mould derivation D_φ reflects the action of the derivation $\operatorname{ad}_{\mathcal{X}_\lambda}$ of $\operatorname{End}_{\boldsymbol{A}}(\mathcal{A})$.

Proof. We first check that, if $\Theta \in \operatorname{End}_{\boldsymbol{A}}(\boldsymbol{A}[[y_1, \dots, y_n]])$ is homogeneous of degree $\eta \in \mathbb{Z}^n$, then

$$[\mathcal{X}_\lambda, \Theta] = \langle \eta, \lambda \rangle \Theta.$$

By $\boldsymbol{A}$-linearity and continuity, it is sufficient to check that both operators act the same way on a monomial y^m. We have $\Theta y^m = \beta_m y^{m+\eta}$ with a $\beta_m \in \boldsymbol{A}$, thus $\mathcal{X}_\lambda \Theta y^m = \langle m+\eta, \lambda\rangle \beta_m y^{m+\eta} = \langle m+\eta, \lambda\rangle \Theta y^m$ while $\Theta \mathcal{X}_\lambda y^m = \langle m, \lambda\rangle \Theta y^m$, hence $[\mathcal{X}_\lambda, \Theta] y^m = \langle \eta, \lambda\rangle \Theta y^m$ as required.

It follows that

$$[\mathcal{X}_\lambda, \boldsymbol{B}_{\boldsymbol{\omega}}] = \big(\varphi(\omega_1) + \dots + \varphi(\omega_r)\big) \boldsymbol{B}_{\boldsymbol{\omega}}, \quad \boldsymbol{\omega} = (\omega_1, \dots, \omega_r) \in \Omega^\bullet.$$

Let $N^\bullet = D_\varphi M^\bullet$. For any exhaustion of $\Omega^\bullet$ by finite sets I_k, letting $\Theta_k = \sum_{\boldsymbol{\omega} \in I_k} M^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}}$ and $\Theta'_k = \sum_{\boldsymbol{\omega} \in I_k} N^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}}$, we get $[\mathcal{X}_\lambda, \Theta_k] = \Theta'_k$. For every $f \in \mathcal{A}$, we have $\Theta'_k f \xrightarrow[k\to\infty]{} \sum N^\bullet \boldsymbol{B}_\bullet f$ on the one hand, while, by continuity of $\mathcal{X}_\lambda$, $[\mathcal{X}_\lambda, \Theta_k] f \xrightarrow[k\to\infty]{} \big[\mathcal{X}_\lambda, \sum M^\bullet \boldsymbol{B}_\bullet\big] f$ on the other hand. □

6.9. Notice that, in the above situation, any $\mathbb{C}$-linear derivation $d \colon \boldsymbol{A} \to \boldsymbol{A}$ induces a derivation $\tilde{d} \colon \mathcal{A} \to \mathcal{A}$ (defined by $\tilde{d} \sum a_m y^m = \sum (d a_m) y^m$) and a mould derivation D (defined just before Remark 4.1). If $\tilde{d}$ commutes with the B_η, $\eta \in \Omega$, one easily gets

$$\big[\tilde{d}, \sum M^\bullet \boldsymbol{B}_\bullet\big] = \sum (D M^\bullet) \boldsymbol{B}_\bullet. \tag{6.6}$$

On the other hand, $D_\varphi M^{\boldsymbol{\omega}} = \langle \|\boldsymbol{\omega}\|, \lambda\rangle M^{\boldsymbol{\omega}}$ if $\varphi = \langle\,\cdot\,, \lambda\rangle$. Thus $D_\varphi = \nabla$ when $n = 1$ and $\lambda = 1$. This is the relevant situation for the saddle-node:

Corollary 6.1. *Choose $\Omega = \mathcal{N}$ as in (3.2), $\boldsymbol{A} = \mathbb{C}[[x]]$ and $(\mathcal{A}, \mathrm{val}) = (\boldsymbol{A}[[y]], \nu_4)$, $\mathcal{F} = \mathcal{F}_{\mathcal{A},\boldsymbol{A}}$. Let $\boldsymbol{B}_\bullet$ denote the $\mathcal{F}$-valued comould generated by $B_\eta = y^{\eta+1} \frac{\partial}{\partial y}$. Let $(a_\eta)_{\eta \in \Omega}$ be as in (3.4) and*

$$X_0 = x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X = X_0 + \sum_{\eta \in \Omega} a_\eta B_\eta.$$

Then the mould-comould contraction $\Theta = \sum \mathcal{V}^\bullet \boldsymbol{B}_\bullet \in \mathcal{F}$, where $\mathcal{V}^\bullet$ is determined by Lemma 3.2, is solution of the conjugacy equation (3.1) $\Theta X = X_0 \Theta$ in $\operatorname{End}_{\mathbb{C}}(\mathcal{A})$.

Proof. It was already observed that each B_η is homogeneous of degree η and the formal summability of $(\mathcal{V}^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \Omega^\bullet}$ was checked in Lemma 3.3. Let $d = x^2 \frac{\mathrm{d}}{\mathrm{d}x} \colon \boldsymbol{A} \to \boldsymbol{A}$.

The corresponding derivation of $\mathcal{A}$ is $\tilde{d} = x^2\frac{\partial}{\partial x}$, which commutes with the B_η's. On the other hand, with the notation of Proposition 6.3, $X_0 = \tilde{d} + \mathcal{X}_1$. Since $X - X_0 = \sum J_a^\bullet \boldsymbol{B}_\bullet$ with the notation of Remark 4.1, equation (3.1) is equivalent to $\big[\tilde{d} + \mathcal{X}_1, \Theta\big] = \Theta \sum J_a^\bullet \boldsymbol{B}_\bullet$; plugging any formally convergent mould-comould expansion $\Theta = \sum M^\bullet \boldsymbol{B}_\bullet$ into it, we find $\sum (DM^\bullet + \nabla M^\bullet)\boldsymbol{B}_\bullet$ for the left-hand side by (6.5) and (6.6) while, according to Proposition 6.1, the right-hand side can be written $\sum (J_a^\bullet \times M^\bullet)\boldsymbol{B}_\bullet$, hence the conclusion follows from (4.5). □

Remark 6.1. The symmetrality of the mould $\mathcal{V}^\bullet$ obtained in Proposition 5.5 shows us that Θ is invertible, with inverse $\Theta^{-1} = \sum \mathcal{V}^\bullet \boldsymbol{B}_\bullet$. The proof of Theorem 1 will thus be complete when we have checked that Θ is an algebra automorphism; this will follow from the results of next section on the contraction of symmetral moulds into a comould generated by derivations.

7 Contraction into a cosymmetral comould

7.1. For the interpretation of alternality and symmetrality of moulds in terms of the corresponding mould-comould expansions, we focus on the case where the comould $\boldsymbol{B}_\bullet$ is generated by a family of $\boldsymbol{A}$-linear derivations $(B_\eta)_{\eta\in\Omega}$ of a commutative algebra $\mathcal{A}$.

The main result of this section is Proposition 7.1 below, according to which, in this case, *the contraction of an alternal mould into* $\boldsymbol{B}_\bullet$ *gives rise to a derivation* and *the contraction of a symmetral mould gives rise to an algebra automorphism.*

7.2. We thus assume that $(\mathcal{A}, \nu)$ is a complete pseudovaluation ring such that $\mathcal{A}$ is a commutative $\boldsymbol{A}$-algebra, and we define $\mathcal{F} = \mathcal{F}_{\mathcal{A},\boldsymbol{A}}$ by (6.2). Since we shall be interested in the way the elements of $\mathcal{F}$ act on products of elements of $\mathcal{A}$, we consider the left $\mathcal{F}$-module $\mathrm{Bil}_{\boldsymbol{A}}$ of $\boldsymbol{A}$-bilinear maps from $\mathcal{A}\times\mathcal{A}$ to $\mathcal{A}$ (with ring multiplication $(\Theta, \Phi) \in \mathcal{F}\times \mathrm{Bil}_{\boldsymbol{A}} \mapsto \Theta\circ\Phi \in \mathrm{Bil}_{\boldsymbol{A}}$) and its filtration

$$\mathcal{B}_\delta = \{\, \Phi \in \mathrm{Bil}_{\boldsymbol{A}} \mid \nu\big(\Phi(f,g)\big) \ge \nu(f) + \nu(g) + \delta \text{ for all } f, g \in \mathcal{A}\,\}, \quad \delta \in \mathbb{Z}.$$

By defining $\mathcal{B} = \bigcup_{\delta\in\mathbb{Z}} \mathcal{B}_\delta$ we get a left $\mathcal{F}$-submodule of $\mathrm{Bil}_{\boldsymbol{A}}$, for which the filtration $(\mathcal{B}_\delta)_{\delta\in\mathbb{Z}}$ is exhaustive, separated and compatible with the filtration of $\mathcal{F}$ induced by val_ν: the subgroups $\mathcal{F}_\delta = \{\, \Theta \in \mathcal{F} \mid \mathrm{val}_\nu(\Theta) \ge \delta \,\}$ satisfy $\mathcal{F}_\delta \mathcal{B}_{\delta'} \subset \mathcal{B}_{\delta+\delta'}$ for all $\delta, \delta' \in \mathbb{Z}$. The corresponding distance on $\mathcal{B}$ is complete, by completeness of $(\mathcal{A}, \nu)$.

We now define a map $\sigma\colon \mathcal{F} \to \mathcal{B}$ by $\Theta \in \mathcal{F} \mapsto \sigma(\Theta) = \Phi$ such that

$$\Phi\colon (f, g) \in \mathcal{A}\times\mathcal{A} \mapsto \Phi(f, g) = \Theta(fg) \in \mathcal{A}.$$

This map is to be understood as a kind of coproduct. Observe that σ is $\mathcal{F}$-linear, i.e. $\sigma(\Theta\Theta') = \Theta\sigma(\Theta')$ (thus it boils down to $\sigma(\Theta) = \Theta\circ\sigma(\mathrm{Id})$, and $\sigma(\mathrm{Id})$ is just the multiplication of $\mathcal{A}$) and continuous because $\sigma(\mathcal{F}_\delta) \subset \mathcal{B}_\delta$ for each $\delta \in \mathbb{Z}$.

Viewing $\mathcal{F}$ as an $\boldsymbol{A}$-module, we also define an $\boldsymbol{A}$-linear map

$$\rho\colon \mathcal{F}_2 = \mathcal{F} \otimes_{\boldsymbol{A}} \mathcal{F} \to \mathcal{B}$$

by its action on decomposable elements:

$$\rho(\Theta_1 \otimes \Theta_2)(f, g) = (\Theta_1 f)(\Theta_2 g), \quad f, g \in \mathcal{A}$$

for any $\Theta_1, \Theta_2 \in \mathcal{F}$. (A remark parallel to the remark on Definition 5.3 applies: the kernel of ρ is the torsion submodule of $\mathcal{F}_2$ when $\mathcal{A}$ is an integral domain; if moreover $\boldsymbol{A}$ is principal, then ρ is injective.) Notice that, for $\Theta_1 \in \mathcal{F}_{\delta_1}$ and $\Theta_2 \in \mathcal{F}_{\delta_2}$, one has $\rho(\Theta_1 \otimes \Theta_2) \in \mathcal{B}_{\delta_1+\delta_2}$, hence the map $\tilde{\rho}\colon (\Theta_1, \Theta_2) \mapsto \rho(\Theta_1 \otimes \Theta_2)$ from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{B}$ is continuous.

Using the $\boldsymbol{A}$-algebra structure of $\mathcal{F}_2$, we see that

$$\sigma(\Theta) = \rho(\xi),\ \sigma(\Theta') = \rho(\xi') \implies \sigma(\Theta\Theta') = \rho(\xi\xi') \tag{7.1}$$

for any $\Theta, \Theta' \in \mathcal{F}$, $\xi, \xi' \in \mathcal{F}_2$.

7.3. With the above notations, the set of all $\boldsymbol{A}$-linear derivations of $\mathcal{A}$ having a valuation is

$$\mathfrak{L}_{\mathcal{F}} = \{\, \Theta \in \mathcal{F} \mid \sigma(\Theta) = \rho(\Theta \otimes \mathrm{Id} + \mathrm{Id} \otimes \Theta) \,\}$$

(it is a Lie algebra for the bracketing $[\Theta_1, \Theta_2] = \Theta_1\Theta_2 - \Theta_2\Theta_1$). Letting $\mathcal{F}^*$ denote the multiplicative group of invertible elements of $\mathcal{F}$, we may also consider its subgroup

$$G_{\mathcal{F}} = \{\, \Theta \in \mathcal{F}^* \mid \sigma(\Theta) = \rho(\Theta \otimes \Theta) \,\},$$

the elements of which are $\boldsymbol{A}$-linear algebra automorphisms of $\mathcal{A}$.

Lemma 7.1. *Assume that the generators B_η, $\eta \in \Omega$, of an $\mathcal{F}$-valued comould $\boldsymbol{B}_\bullet$ all belong to $\mathfrak{L}_{\mathcal{F}}$. Then*

$$\sigma(\boldsymbol{B}_{\boldsymbol{\omega}}) = \sum_{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \in \Omega^\bullet} \operatorname{sh}\begin{pmatrix} \boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \\ \boldsymbol{\omega} \end{pmatrix} \rho(\boldsymbol{B}_{\boldsymbol{\omega}^1} \otimes \boldsymbol{B}_{\boldsymbol{\omega}^2}), \quad \boldsymbol{\omega} \in \Omega^\bullet. \tag{7.2}$$

Such a comould is said to be *cosymmetral*.

Proof. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r) \in \Omega^\bullet$. We proceed by induction on r. Equation (7.2) holds if $r = 0$, since $\sigma(\mathrm{Id}) = \rho(\mathrm{Id} \otimes \mathrm{Id})$, or $r = 1$ (by assumption); we thus suppose $r \geq 2$. By (6.1), we can write $\boldsymbol{B}_{\boldsymbol{\omega}} = \boldsymbol{B}_{`\boldsymbol{\omega}} B_{\omega_1}$ with $`\boldsymbol{\omega} = (\omega_2, \dots, \omega_r)$. Using the induction hypothesis and (7.1), we get $\sigma(\boldsymbol{B}_{\boldsymbol{\omega}}) = \rho(\xi)$ with

$$\begin{aligned}
\xi &= \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega^\bullet} \operatorname{sh}\begin{pmatrix} \boldsymbol{\alpha}, \boldsymbol{\beta} \\ `\boldsymbol{\omega} \end{pmatrix} (\boldsymbol{B}_{\boldsymbol{\alpha}} \otimes \boldsymbol{B}_{\boldsymbol{\beta}})(B_{\omega_1} \otimes \mathrm{Id} + \mathrm{Id} \otimes B_{\omega_1}) \\
&= \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega^\bullet} \operatorname{sh}\begin{pmatrix} \boldsymbol{\alpha}, \boldsymbol{\beta} \\ `\boldsymbol{\omega} \end{pmatrix} (\boldsymbol{B}_{\omega_1 \centerdot \boldsymbol{\alpha}} \otimes \boldsymbol{B}_{\boldsymbol{\beta}} + \boldsymbol{B}_{\boldsymbol{\alpha}} \otimes \boldsymbol{B}_{\omega_1 \centerdot \boldsymbol{\beta}}).
\end{aligned}$$

This coincides with $\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\Omega^\bullet} \operatorname{sh}\binom{\boldsymbol{\alpha},\,\boldsymbol{\beta}}{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\alpha}} \otimes \boldsymbol{B}_{\boldsymbol{\beta}}$, since

$$\operatorname{sh}\begin{pmatrix}\boldsymbol{\alpha},\,\boldsymbol{\beta}\\ \boldsymbol{\omega}\end{pmatrix} = \operatorname{sh}\begin{pmatrix}`\boldsymbol{\alpha},\,\boldsymbol{\beta}\\ `\boldsymbol{\omega}\end{pmatrix} 1_{\{\alpha_1=\omega_1\}} + \operatorname{sh}\begin{pmatrix}\boldsymbol{\alpha},\,`\boldsymbol{\beta}\\ `\boldsymbol{\omega}\end{pmatrix} 1_{\{\beta_1=\omega_1\}}$$

(particular case of (5.6) with $\boldsymbol{\gamma}^1 = \omega_1$ and $\boldsymbol{\gamma}^2 = `\boldsymbol{\omega}$). □

7.4. We are now ready to study the effect of alternality or symmetrality in this context.

Proposition 7.1. *Suppose that $\boldsymbol{B}_\bullet$ is an $\mathcal{F}$-valued cosymmetral comould and let $M^\bullet \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ be such that $(M^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet}$ is formally summable. Then:*

- *If $M^\bullet$ is alternal, then $\sum M^\bullet \boldsymbol{B}_\bullet \in \mathfrak{L}_{\mathcal{F}}$.*
- *If $M^\bullet$ is symmetral, then $\sum M^\bullet \boldsymbol{B}_\bullet \in G_{\mathcal{F}}$.*
- *More generally, denoting by $\boldsymbol{M}^{\bullet,\bullet}$ the image of $M^\bullet$ by the homomorphism τ of Lemma 5.1 and assuming that the family $\big(\rho(\boldsymbol{M}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \boldsymbol{B}_{\boldsymbol{\alpha}} \otimes \boldsymbol{B}_{\boldsymbol{\beta}})\big)_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Omega^\bullet\times\Omega^\bullet}$ is formally summable in $\mathcal{B}$,*

$$\sigma\Big(\sum M^\bullet \boldsymbol{B}_\bullet\Big) = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Omega^\bullet\times\Omega^\bullet} \rho(\boldsymbol{M}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \boldsymbol{B}_{\boldsymbol{\alpha}} \otimes \boldsymbol{B}_{\boldsymbol{\beta}}). \tag{7.3}$$

Proof. Let $\delta_* \in \mathbb{Z}$ such that $v(\boldsymbol{\omega}) = \operatorname{val}_\nu(M^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}}) \geq \delta_*$ for all $\boldsymbol{\omega} \in \Omega^\bullet$ and $\Theta = \sum M^\bullet \boldsymbol{B}_\bullet$. We shall use the notation $\Phi_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \boldsymbol{M}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \boldsymbol{B}_{\boldsymbol{\alpha}} \otimes \boldsymbol{B}_{\boldsymbol{\beta}}$ for $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \Omega^\bullet \times \Omega^\bullet$. Lemma 5.2 yields

$$\Phi_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 1_{\{\boldsymbol{\beta}=\emptyset\}} M^{\boldsymbol{\alpha}} \boldsymbol{B}_{\boldsymbol{\alpha}} \otimes \operatorname{Id} + 1_{\{\boldsymbol{\alpha}=\emptyset\}} \operatorname{Id} \otimes M^{\boldsymbol{\beta}} \boldsymbol{B}_{\boldsymbol{\beta}}$$

for $M^\bullet$ alternal and $\Phi_{\boldsymbol{\alpha},\boldsymbol{\beta}} = M^{\boldsymbol{\alpha}} \boldsymbol{B}_{\boldsymbol{\alpha}} \otimes M^{\boldsymbol{\beta}} \boldsymbol{B}_{\boldsymbol{\beta}}$ for $M^\bullet$ symmetral. In both cases, the set $\{ (\boldsymbol{\alpha},\boldsymbol{\beta}) \in \Omega^\bullet \times \Omega^\bullet \mid \rho(\Phi_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \notin \mathcal{B}_\delta \}$ is thus finite for any $\delta \in \mathbb{Z}$, in view of the formal summability hypothesis, and the sum of the family $\big(\rho(\Phi_{\boldsymbol{\alpha},\boldsymbol{\beta}})\big)$ is respectively $\rho(\Theta \otimes \operatorname{Id} + \operatorname{Id} \otimes \Theta)$ or $\rho(\Theta \otimes \Theta)$, by continuity of $\tilde{\rho}$. Therefore it is sufficient to prove the third property (the invertibility of Θ when $M^\bullet$ is symmetral is a simple consequence of Proposition 6.1 and of the invertibility of $M^\bullet$).

We thus assume $\big(\rho(\Phi_{\boldsymbol{\alpha},\boldsymbol{\beta}})\big)$ formally summable in $\mathcal{B}$. As in the proof of Proposition 6.1, we can suppose Ω countable and choose an exhaustion of $\Omega^\bullet$ by finite sets of the form $\Omega^{K,R}$. Then, by virtue of the definition of τ and of Lemma 7.1,

$$\begin{aligned} A_{K,R} &:= \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Omega^{K,R}\times\Omega^{K,R}} \rho(\Phi_{\boldsymbol{\alpha},\boldsymbol{\beta}}) - \sigma\Big(\sum_{\boldsymbol{\omega}\in\Omega^{K,R}} M^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}}\Big) \\ &= \Big(\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\Omega^{K,R}\\ \boldsymbol{\omega}\in\Omega^\bullet}} - \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\Omega^\bullet\\ \boldsymbol{\omega}\in\Omega^{K,R}}}\Big) \operatorname{sh}\begin{pmatrix}\boldsymbol{\alpha},\,\boldsymbol{\beta}\\ \boldsymbol{\omega}\end{pmatrix} M^{\boldsymbol{\omega}} \rho(\boldsymbol{B}_{\boldsymbol{\alpha}} \otimes \boldsymbol{B}_{\boldsymbol{\beta}}) \\ &= \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\Omega^{K,R}\\ \text{s.t. } r(\boldsymbol{\alpha})+r(\boldsymbol{\beta})>R}} \rho(\Phi_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \end{aligned}$$

(the last equality stems from the fact that, if $\boldsymbol{\omega} \in \Omega^{K,R}$, then $\operatorname{sh}\binom{\boldsymbol{\alpha},\,\boldsymbol{\beta}}{\boldsymbol{\omega}} \neq 0$ implies $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega^{K,R}$). The formal summability of $\big(\rho(\Phi_{\boldsymbol{\alpha},\boldsymbol{\beta}})\big)$ yields $A_{K,R} \to 0$ as $K, R \to \infty$, which is the desired result since σ is continuous. $\square$

Remark 7.1. The proof of Theorem 1 is now complete: in view of the symmetrality of $\mathcal{V}^\bullet$ with $\Omega = \mathcal{N}$ and $\boldsymbol{A} = \mathbb{C}[[x]]$ (Proposition 5.5) and the cosymmetrality of $\boldsymbol{B}_\bullet$ defined by (3.5) with $\mathcal{F} = \mathcal{F}_{\mathcal{A},\boldsymbol{A}}$, $(\mathcal{A}, \operatorname{val}) = (\mathbb{C}[[x, y]], \nu_4)$, Proposition 7.1 shows that $\Theta = \sum \mathcal{V}^\bullet \boldsymbol{B}_\bullet$ is an automorphism of $\mathcal{A}$. As noticed in Remark 6.1, this was the only thing which remained to be checked.

7.5. Another way of checking that the contraction of an alternal mould into a cosymmetral comould $\boldsymbol{B}_\bullet$ is a derivation is to express it as a sum of iterated Lie brackets of the derivations B_η which generate the comould.

For $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r) \in \Omega^\bullet$ with $r \geq 2$, let

$$\boldsymbol{B}_{[\boldsymbol{\omega}]} = [B_{\omega_r}, [B_{\omega_{r-1}}, [\cdots [B_{\omega_2}, B_{\omega_1}] \cdots]]].$$

One can check that, for any alternal mould $M^\bullet$ and for any finite subset Ω_f of Ω,

$$\sum_{\boldsymbol{\omega} \in \Omega_f^r} M^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}} = \frac{1}{r} \sum_{\boldsymbol{\omega} \in \Omega_f^r} M^{\boldsymbol{\omega}} \boldsymbol{B}_{[\boldsymbol{\omega}]}, \quad r \geq 2$$

(identifying Ω_f^r with the sets of all words of length r the letters of which belong to Ω_f). The proof is left to the reader.

7.6. Let $\boldsymbol{B}_\bullet$ denote an $\mathcal{F}$-valued comould. Suppose that $(B_\eta)_{\eta \in \Omega}$ is formally summable and consider $Y = \sum_{\eta \in \Omega} B_\eta \in \mathcal{F}$.

We have seen that, by definition, the comould is cosymmetral iff each B_η is a derivation of $\mathcal{A}$; then Y is itself a derivation. This is the situation when there is an appropriate notion of homogeneity, as in Definition 6.2, and we expand an $\boldsymbol{A}$-linear derivation Y into a sum of homogeneous components, each B_η being homogeneous of degree η.

Suppose now that the object to analyse is not a singular vector field, as in the case of the saddle-node, but a local transformation; considering the associated substitution operator, we are thus led to an automorphism of $\mathcal{A}$, typically of the form $\phi = \operatorname{Id} + Y$. Then the homogeneous components B_η of Y are no longer derivations; expanding $\sigma(\phi) = \rho(\phi \otimes \phi)$, we rather get

$$\sigma(B_\eta) = \rho\Big(B_\eta \otimes \operatorname{Id} + \sum_{\eta' + \eta'' = \eta} B_{\eta'} \otimes B_{\eta''} + \operatorname{Id} \otimes B_\eta\Big). \tag{7.4}$$

The comould $\boldsymbol{B}_\bullet$ they generate is then called *cosymmetrel*. A cosymmetrel comould is characterized by identities similar to (7.2) but with the shuffling coefficients $\operatorname{sh}\binom{\boldsymbol{\omega}^1,\,\boldsymbol{\omega}^2}{\boldsymbol{\omega}}$ replaced by new ones, denoted by $\operatorname{ctsh}\binom{\boldsymbol{\omega}^1,\,\boldsymbol{\omega}^2}{\boldsymbol{\omega}}$ and called "contracting shuffling coefficients".

Dually, using these new coefficients instead of the previous shuffling coefficients in formulas (5.1) and (5.2), one gets the definition of *alternel* and *symmetrel* moulds, which were only briefly alluded to at the beginning of Section 5.

The contraction of alternel or symmetrel moulds into cosymmetrel comoulds enjoy properties parallel to those that we just described in the cosymmetral case. This allows one to treat local vector fields and local discrete dynamical systems with completely parallel formalisms.

C Resurgence, alien calculus and other applications

8 Resurgence of the normalising series

8.1. The purpose of this section is to use the mould-comould representation of the formal normalisation of the saddle-node given by Theorem 1 to deduce "resurgent properties". We begin by a few reminders about Écalle's resurgence theory. We follow the notations of [14].

– The formal Borel transform is the $\mathbb{C}$-linear homomorphism

$$\mathcal{B} \;:\; \widetilde{\varphi}(z) = \sum_{n\geq 0} c_n z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]] \;\mapsto\; \widehat{\varphi}(\zeta) = \sum_{n\geq 0} c_n \frac{\zeta^n}{n!} \in \mathbb{C}[[\zeta]].$$

In the case of a convergent $\widetilde{\varphi}$, one gets a convergent series $\widehat{\varphi}$ which defines an entire function of exponential type. Namely, if $\varphi(x) = \widetilde{\varphi}(-1/x) \in x\mathbb{C}[[x]]$ has radius of convergence $> \rho$, then there exists $K > 0$ such that $|\varphi(x)| \leq K|x|$ for $|x| \leq \rho$ and this implies, by virtue of the Cauchy inequalities, that $|c_n| \leq K\rho^{-n}$, hence $\widehat{\varphi}$ is entire and

$$|\widehat{\varphi}(\zeta)| \leq K\,\mathrm{e}^{\rho^{-1}|\zeta|}, \quad \zeta \in \mathbb{C}. \tag{8.1}$$

We are particularly interested in the case where $\widehat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ without being necessarily entire; this is equivalent to Gevrey-1 growth for the coefficients of $\widetilde{\varphi}$:

$$\begin{aligned} \widetilde{\varphi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]_1 \;&\overset{\text{def}}{\iff}\; \text{there are } C, K > 0 \text{ such that } |c_n| \leq KC^n n! \text{ for all } n \\ &\iff\; \mathcal{B}\widetilde{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}. \end{aligned}$$

– The counterpart in $\mathbb{C}[[\zeta]]$ of the multiplication (Cauchy product) of $z^{-1}\mathbb{C}[[z^{-1}]]$ is called *convolution* and denoted by $*$, thus $\mathcal{B}(\widetilde{\varphi}\cdot\widetilde{\psi}) = \mathcal{B}(\widetilde{\varphi}) * \mathcal{B}(\widetilde{\psi})$. Now, if $\widehat{\varphi} = \mathcal{B}(\widetilde{\varphi})$ and $\widehat{\psi} = \mathcal{B}(\widetilde{\psi})$ belong to $\mathbb{C}\{\zeta\}$, then $\widehat{\varphi} * \widehat{\psi} \in \mathbb{C}\{\zeta\}$ and this germ of holomorphic function is determined by

$$(\widehat{\varphi} * \widehat{\psi})(\zeta) = \int_0^{\zeta} \widehat{\varphi}(\zeta_1)\widehat{\psi}(\zeta - \zeta_1) \quad \text{for } |\zeta| \text{ small enough.} \tag{8.2}$$

We have an algebra $(\mathbb{C}\{\zeta\}, *)$ without unit, isomorphic via $\mathcal{B}$ to $(z^{-1}\mathbb{C}[[z^{-1}]]_1, \cdot)$. By adjunction of unit, we get an algebra isomorphism

$$\mathcal{B}\colon \mathbb{C}[[z^{-1}]]_1 \xrightarrow{\sim} \mathbb{C}\,\delta \oplus \mathbb{C}\{\zeta\}$$

(where $\delta = \mathcal{B}1$ is a symbol for the unit of convolution). We can even take into account the differential $\frac{\mathrm{d}}{\mathrm{d}z}$: its counterpart via $\mathcal{B}$ is $\widehat{\partial}\colon c\,\delta + \widehat{\varphi}(\zeta) \mapsto -\zeta\widehat{\varphi}(\zeta)$.

– Let us now consider all the rectifiable oriented paths of $\mathbb{C}$ which start from the origin and then avoid $\mathbb{Z}$, i.e. oriented paths represented by absolutely continuous maps $\gamma\colon [0,1] \to \mathbb{C} \setminus \mathbb{Z}^*$ such that $\gamma(0) = 0$ and $\gamma^{-1}(0)$ is connected. We denote by $\mathcal{R}(\mathbb{Z})$ the set of all homotopy classes $[\gamma]$ of such paths γ and by π the map $[\gamma] \in \mathcal{R}(\mathbb{Z}) \mapsto \gamma(1) \in \mathbb{C} \setminus \mathbb{Z}^*$; considering π as a covering map, we get a Riemann surface structure on $\mathcal{R}(\mathbb{Z})$.

Observe that $\pi^{-1}(0)$ consists of a single point, the "origin" of $\mathcal{R}(\mathbb{Z})$; this is the only difference between $\mathcal{R}(\mathbb{Z})$ and the universal cover of $\mathbb{C} \setminus \mathbb{Z}$. The space $\widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ of all holomorphic functions of $\mathcal{R}(\mathbb{Z})$ can be identified with the space of all $\widehat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ which admit an analytic continuation along any representative of any element of $\mathcal{R}(\mathbb{Z})$ (cf. [14], Definition 3 and Lemma 2).

Definition 8.1. We define the convolutive model of the algebra of resurgent functions over $\mathbb{Z}$ as $\widehat{\boldsymbol{R}}_{\mathbb{Z}} = \mathbb{C}\,\delta \oplus \widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$. We define the formal model of the algebra of resurgent functions over $\mathbb{Z}$ as $\widetilde{\boldsymbol{R}}_{\mathbb{Z}} = \mathcal{B}^{-1}(\widehat{\boldsymbol{R}}_{\mathbb{Z}})$.

It turns out that $\widehat{\boldsymbol{R}}_{\mathbb{Z}}$ is a subalgebra of the convolution algebra $\mathbb{C}\,\delta \oplus \mathbb{C}\{\zeta\}$, i.e. the aforementioned property of analytic continuation is stable by convolution; the proof of this fact relies on the notion of symmetrically contractile path (see for instance *op. cit.*, §1.3), which we shall not develop here. Therefore $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$ is a subalgebra of $\mathbb{C}[[z^{-1}]]$ and $\mathcal{B}$ induces an algebra isomorphism $\widetilde{\boldsymbol{R}}_{\mathbb{Z}} \to \widehat{\boldsymbol{R}}_{\mathbb{Z}}$.

– An obvious example of element of $\widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ is an entire function, or a meromorphic function of $\mathbb{C}$ without poles outside $\mathbb{Z}^*$. Indeed, for such a function $\widehat{\varphi}$, we can define $\hat{\phi} \in \widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ by $\hat{\phi}(\zeta) = \widehat{\varphi}\big(\pi(\zeta)\big)$ for all $\zeta \in \mathcal{R}(\mathbb{Z})$. We usually identify $\widehat{\varphi}$ and $\hat{\phi}$.

For example, if $\omega_1 \neq 0$, Remark 3.1 shows that $\widetilde{\mathcal{V}}^{\omega_1}(z) = \mathcal{V}^{\omega_1}(-1/z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ has formal Borel transform $\widehat{\mathcal{V}}^{\omega_1}(\zeta) = \sum \omega_1^{-r-1}(-\widehat{\partial})^r \hat{a}_{\omega_1} = \frac{\hat{a}_{\omega_1}(\zeta)}{\omega_1-\zeta}$, where $\hat{a}_{\omega_1}$ denotes the formal Borel transform of $a_{\omega_1}(-1/z)$, which is an entire function (since a_{ω_1} is convergent), thus $\widehat{\mathcal{V}}^{\omega_1}$ is meromorphic with at most one simple pole, located at ω_1. On the other hand, $\widehat{\mathcal{V}}^0(\zeta) = \frac{1}{\zeta}\hat{a}_0(\zeta)$ is entire.

We shall see that, for each non-empty word $\boldsymbol{\omega}$, the formal Borel transform of $\mathcal{V}^{\boldsymbol{\omega}}(-1/z)$ belongs to $\widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$, but this function is usually not meromorphic if $r(\boldsymbol{\omega}) \geq 2$. For instance, for $\boldsymbol{\omega} = (\omega_1, \omega_2)$, one gets $\frac{1}{-\zeta+\omega_1+\omega_2}\big(\hat{a}_{\omega_1} * \widehat{\mathcal{V}}^{\omega_2}\big)$ which is multivalued in general (see formula (8.9) below for the general case).

– A formal series $\widetilde{\varphi}(z)$ without constant term belongs to $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$ iff its formal Borel transform $\widehat{\varphi}(\zeta)$ converges to a germ of holomorphic function which extends analytically to $\mathcal{R}(\mathbb{Z})$. In particular, the principal branch[11] of $\widehat{\varphi}$ is holomorphic in sectors which extend up to infinity. If it has at most exponential growth in a sector $\{\,\zeta \in \mathbb{C} \mid \theta_1 \le \arg\zeta \le \theta_2\,\}$ (as is the case of $\widehat{\mathcal{V}}^{\omega_1}(\zeta)$ for instance), then one can perform a Laplace transform and get a function

$$\widetilde{\varphi}^{\mathrm{ana}}(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \widehat{\varphi}(\zeta)\,\mathrm{e}^{-z\zeta}\,\mathrm{d}\zeta, \quad \theta \in [\theta_1, \theta_2],$$

which is analytic for z belonging to a sectorial neighbourhood of infinity. This is called *Borel–Laplace summation* (see e.g. [14], §1.1).

Since multiplication is turned into convolution by $\mathcal{B}$ and then turned again into multiplication by the Laplace transform, and similarly with $\frac{\mathrm{d}}{\mathrm{d}z}$ which is transformed into multiplication by $-\zeta$ by $\mathcal{B}$, the Borel–Laplace process transforms the formal solution of a differential equation like (2.8), (2.9) or (3.7) into an analytic solution of the same equation.

8.2. The stability of $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$ under multiplication together with the previous computation explains to some extent why we can expect the solutions of a non-linear problem like the formal classification of the saddle-node to be resurgent. However, controlling products in $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$ means controlling convolution products in $\widehat{\boldsymbol{R}}_{\mathbb{Z}}$, and it is not so easy to extract from the stability statement the quantitative information which would guarantee the convergence in $\widehat{\boldsymbol{R}}_{\mathbb{Z}}$ of a method of majorant series for instance (see the discussion at the end of the sketch of proof of Theorem 2 of [14]).

Thanks to the mould-comould expansion given in Section 3, we shall be able to use much simpler arguments: the convolution product of an element of $\widehat{\boldsymbol{R}}_{\mathbb{Z}}$ with an entire function belongs to $\widehat{\boldsymbol{R}}_{\mathbb{Z}}$ and efficient bounds are available in this particular case of the stability statement (much easier than the general one – see Lemma 8.3 below).

Theorem 2. *Consider the saddle-node problem, with hypotheses* (2.1)–(2.2). *Let* $\theta(x,y) = \big(x, y + \sum_{n\ge 0} \varphi_n(x) y^n\big)$ *denote the formal transformation, the substitution operator of which is* $\Theta = \sum \mathcal{V}^{\bullet}\boldsymbol{B}_{\bullet}$, *in accordance with Lemmas* 3.2 *and* 3.3 *and Theorem* 1. *Let* $\theta^{-1}(x,y) = \big(x, y + \sum_{n\ge 0} \psi_n(x) y^n\big)$ *denote the inverse transformation.*

Then, for each $n \in \mathbb{N}$, *the formal series* $\widetilde{\varphi}_n(z) = \varphi_n(-1/z)$ *and* $\widetilde{\psi}_n(z) = \psi_n(-1/z)$ *belong to* $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$, *and the analytic continuation of the formal Borel transforms* $\widehat{\varphi}_n(\zeta), \widehat{\psi}_n(\zeta) \in \mathbb{C}\{\zeta\}$ *satisfy the following:*

(i) *All the branches of the analytic continuation of* $\widehat{\varphi}_n$ *are regular at the points of* $n + \mathbb{N} = \{n, n+1, n+2, \dots\}$.

(ii) *All the branches of the analytic continuation of* $\widehat{\psi}_n$ *are regular at the points of* $-\mathbb{N}^* = \{-1, -2, -3 \dots\}$, *with the sole exception that the branches of* $\widehat{\psi}_0$ *may have simple poles at* -1.

[11] The principal branch is defined as the analytic continuation of $\widehat{\varphi}$ in the maximal open subset of $\mathbb{C}$ which is star-shaped with respect to 0; its domain is the cut plane obtained by removing the singular half-lines $[1,+\infty[$ and $]-\infty,-1]$, unless $\widehat{\varphi}$ happens to be regular at 1 or -1.

(iii) *Given any* $\rho \in \left]0, \frac{1}{2}\right[$ *and* $N \in \mathbb{N}^*$*, there exist positive constants* K, L, C *which depend only on* ρ, N *such that, for any* $(\rho, N, n - \mathbb{N}^*)$*-adapted infinite path* γ *issuing from the origin,*

$$\left|\widehat{\varphi}_n\big(\gamma(t)\big)\right| \le K L^n \,\mathrm{e}^{(n^2+1)Ct} \quad \textit{for all } t \ge 0 \textit{ and } n \in \mathbb{N}, \tag{8.3}$$

while, for any $(\rho, N, \mathbb{N})$*-adapted infinite path* γ *issuing from the origin,*

$$\left|\widehat{\psi}_n\big(\gamma(t)\big)\right| \le K L^n \,\mathrm{e}^{Ct} \ \textit{for } n \ge 1, \quad \left|\big(\gamma(t)+1\big)\widehat{\psi}_0\big(\gamma(t)\big)\right| \le K \,\mathrm{e}^{Ct} \tag{8.4}$$

for all $t \ge 0$.

What we call $(\rho, N, \mathcal{P}^\pm)$-adapted infinite path, with $\mathcal{P}^+ = \mathbb{N}$ or $\mathcal{P}^- = n - \mathbb{N}^*$, is defined below in Definition 8.2; see Figure 1 on p. 125. These are arc-length parametrised paths $\gamma\colon\ [0, +\infty[\to \mathbb{C}$ (i.e. γ is absolutely continuous and $|\dot\gamma(t)| = 1$ for almost every t) which start as rectilinear segments of length ρ issuing from the origin and which then do not approach $\mathcal{P}^\pm$ nor $\pm\Sigma(\rho, N)$ at a distance $< \rho$, where $\pm\Sigma(\rho, N)$ denotes the sector of half-opening $\arcsin(\rho/N)$ bisected by $\pm\,[N, +\infty[$.

In particular, inequalities (8.3)–(8.4) yield an exponential bound at infinity for the principal branch of each $\widehat{\varphi}_n$ or $\widehat{\psi}_n$ along all the half-lines issuing from 0 except the singular half-lines $\pm\,[0, +\infty[$ (the half-line $[0, +\infty[$ is not singular for $\widehat{\varphi}_0$ and the half-line $-\,[0, +\infty[$ is not singular for any $\widehat{\psi}_n$ with $n \ge 1$).

We recall that Θ establishes a conjugacy between the saddle-node vector field X and its normal form X_0, thus the formal series $\widetilde{\varphi}_n(z)$ are the components of a formal integral $\widetilde{Y}(z, u)$, as described in (2.6)–(2.7). The resurgence statement contained in Theorem 2 thus means that the formal solutions of the singular differential equations (2.8)–(2.9) may be divergent but that this divergence is of a very precise nature. We shall briefly indicate in Section 10 how alien calculus allows one to take advantage of this information to study the problem of analytic classification.

Remark 8.1. Theorem 2 also permits the obtention of analytic solutions of (2.8)–(2.9) via Borel–Laplace summation. It is thus worth mentioning that one can get rid of the dependence on n in the exponential which appears in (8.3), provided one restricts oneself to paths which start from the origin and then do not approach at a distance $< \rho$ the set $\mathbb{Z} \cup \Sigma(\rho, N) \cup \big(-\Sigma(\rho, N)\big)$, and which cross the cuts (the segments between consecutive points of $\mathbb{Z}$) at most N' times. For instance, with $N' = 0$, one obtains

$$\left|\widehat{\varphi}_n(\zeta)\right| \le K L^n \,\mathrm{e}^{C|\zeta|} \tag{8.5}$$

for the principal branch of $\widehat{\varphi}_n$, possibly with larger constants K, L, C but still independent of n. For the other branches, which correspond to $N' \ge 1$ and $N \ge 2$, one has to resort to symmetrically contractile paths and the implied constants K, L, C depend only on ρ, N, N'.

Therefore, when performing Laplace transform, inequalities (8.5) allow one to get the same domain of analyticity for all the functions $\widetilde{\varphi}_n^{\mathrm{ana}}(z)$ solutions of (2.8)–(2.9) (a sectorial neighbourhood of infinity which depends only on C; see e.g. [14],

§1.1), with explicit bounds which make it possible to study the domain of analyticity of a *sectorial formal integral* $\widetilde{Y}^{\mathrm{ana}}(z,u) = u\,\mathrm{e}^z + \sum u^n\,\mathrm{e}^{nz}\widetilde{\varphi}_n^{\mathrm{ana}}(z)$ or of analytic normalising transformations $\varphi^{\mathrm{ana}}(x,y)$, $\psi^{\mathrm{ana}}(x,y)$.

This will be used in Section 11.

The rest of this section is devoted to the proof of Theorem 2 and to the derivation of inequalities (8.5).

8.3. Using $\Omega = \mathcal{N} = \{\eta \in \mathbb{Z} \mid \eta \geq -1\}$ as an alphabet, we know that the $\mathbb{C}[[x]]$-valued moulds $\mathcal{V}^\bullet$ and $\mathcal{V\!\!\!/}^\bullet$ are symmetral and mutually inverse for mould multiplication. We recall that

$$\Theta^{-1} = \sum \mathcal{V\!\!\!/}^\bullet \boldsymbol{B}_\bullet, \qquad \mathcal{V\!\!\!/}^{\omega_1,\dots,\omega_r} = (-1)^r\,\mathcal{V}^{\omega_r,\dots,\omega_1}.$$

With the notation $\|\boldsymbol{\omega}\| = \omega_1 + \cdots + \omega_r$ for any non-empty word $\boldsymbol{\omega} \in \Omega^\bullet$, equation (3.11) can be written $\boldsymbol{B}_{\boldsymbol{\omega}} y = \beta_{\boldsymbol{\omega}} y^{\|\boldsymbol{\omega}\|+1}$, with the coefficients $\beta_{\boldsymbol{\omega}}$ defined at the end of Section 3. As was already observed, since $\Theta y = \sum \varphi_n(x) y^n$ and $\Theta^{-1} y = \sum \psi_n(x) y^n$, the formal series we are interested in can be written as formally convergent series in $\mathbb{C}[[x]]$:

$$\varphi_n = \sum_{\|\boldsymbol{\omega}\|=n-1} \beta_{\boldsymbol{\omega}} \mathcal{V}^{\boldsymbol{\omega}}, \qquad \psi_n = \sum_{\|\boldsymbol{\omega}\|=n-1} \beta_{\boldsymbol{\omega}} \mathcal{V\!\!\!/}^{\boldsymbol{\omega}}, \qquad n \in \mathbb{N}, \tag{8.6}$$

with summation over all words $\boldsymbol{\omega}$ of positive length subject to the condition $\|\boldsymbol{\omega}\| = n-1$. In fact, not all of these words contribute in these series:

Lemma 8.1. *For any non-empty* $\boldsymbol{\omega} = (\omega_1,\dots,\omega_r) \in \Omega^\bullet$*, using the notations*

$$\check{\omega}_i = \omega_1 + \cdots + \omega_i, \quad \hat{\omega}_i = \omega_i + \cdots + \omega_r, \quad 1 \leq i \leq r, \tag{8.7}$$

we have

$$\beta_{\boldsymbol{\omega}} \neq 0 \implies \|\boldsymbol{\omega}\| \geq -1,\ \check{\omega}_1,\dots,\check{\omega}_{r-1} \geq 0 \ \textit{and}\ \hat{\omega}_1,\dots,\hat{\omega}_r \leq \|\boldsymbol{\omega}\|.$$

Proof. We have

$$\beta_{\boldsymbol{\omega}} = 1 \ \text{ if } r = 1, \quad \beta_{\boldsymbol{\omega}} = (\check{\omega}_1 + 1)(\check{\omega}_2 + 1)\cdots(\check{\omega}_{r-1} + 1) \ \text{ if } r \geq 2. \tag{8.8}$$

The property $\beta_{\boldsymbol{\omega}} \neq 0 \implies \|\boldsymbol{\omega}\| \geq -1$ was already observed at the end of Section 3, as a consequence of $\boldsymbol{B}_{\boldsymbol{\omega}} y \in \mathbb{C}[[y]]$ (one can also argue directly from formula (8.8)).

Now suppose $\beta_{\boldsymbol{\omega}} \neq 0$ and $1 \leq i \leq r-1$. The identity

$$\beta_{\boldsymbol{\omega}} = \beta_{\omega_1,\dots,\omega_i}(\check{\omega}_i + 1)\cdots(\check{\omega}_{r-1} + 1)$$

implies $\check{\omega}_i \neq -1$ and $\beta_{\omega_1,\dots,\omega_i} \neq 0$, hence $\omega_1 + \cdots + \omega_i \geq -1$. Therefore $\check{\omega}_i \geq 0$ and $\hat{\omega}_{i+1} = \|\boldsymbol{\omega}\| - \check{\omega}_i \leq \|\boldsymbol{\omega}\|$, while $\hat{\omega}_1 = \|\boldsymbol{\omega}\|$. □

8.4. We recall that the convergent series $a_\eta(x)$ were defined in (3.2) as Taylor coefficients with respect to y of the saddle-node vector field (2.1). We define $\widetilde{\varphi}_n(z)$, $\widetilde{\psi}_n(z)$,

$\tilde{a}_\eta(z)$, $\tilde{\mathcal{V}}^{\boldsymbol{\omega}}(z)$, $\tilde{\mathcal{V}}\!\!\!/^{\boldsymbol{\omega}}(z)$ from $\varphi_n(x)$, $\psi_n(x)$, $a_\eta(x)$, $\mathcal{V}^{\boldsymbol{\omega}}(x)$, $\mathcal{V}\!\!\!/^{\boldsymbol{\omega}}(x)$ by the change of variable $z = -1/x$ (for any $n \in \mathbb{N}$, $\eta \in \Omega$, $\boldsymbol{\omega} \in \Omega^\bullet$), and we denote by $\hat{\varphi}_n(\zeta)$, $\hat{\psi}_n(\zeta)$, etc. the formal Borel transforms of these formal series.

In view of Lemma 3.2, the formal series $\tilde{\mathcal{V}}^{\boldsymbol{\omega}}$ are uniquely determined by the equations $\tilde{\mathcal{V}}^{\emptyset} = 1$ and

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + \|\boldsymbol{\omega}\|\right)\tilde{\mathcal{V}}^{\boldsymbol{\omega}} = \tilde{a}_{\omega_1}\tilde{\mathcal{V}}^{`\boldsymbol{\omega}}, \quad \tilde{\mathcal{V}}^{\boldsymbol{\omega}} \in z^{-1}\mathbb{C}[[z^{-1}]]$$

for $\boldsymbol{\omega}$ non-empty, with $`\boldsymbol{\omega}$ denoting $\boldsymbol{\omega}$ deprived from its first letter. Since $\mathcal{B}$ transforms $\frac{\mathrm{d}}{\mathrm{d}z}$ into multiplication by $-\zeta$ and multiplication into convolution, we get $\hat{\mathcal{V}}^{\emptyset} = \delta$ and

$$\hat{\mathcal{V}}^{\boldsymbol{\omega}}(\zeta) = -\frac{1}{\zeta - \|\boldsymbol{\omega}\|}\big(\hat{a}_{\omega_1} * \hat{\mathcal{V}}^{`\boldsymbol{\omega}}\big), \quad \boldsymbol{\omega} \neq \emptyset,$$

where the right-hand side belongs to $\mathbb{C}[[\zeta]]$ even if $\|\boldsymbol{\omega}\| = 0$, by the same argument as in the proof of Lemma 3.2. It belongs in fact to $\mathbb{C}\{\zeta\}$, by induction on $r(\boldsymbol{\omega})$, and

$$\hat{\mathcal{V}}^{\boldsymbol{\omega}} = (-1)^r \frac{1}{\zeta - \hat{\omega}_1}\left(\hat{a}_{\omega_1} * \left(\frac{1}{\zeta - \hat{\omega}_2}\left(\hat{a}_{\omega_2} * \left(\cdots\left(\frac{1}{\zeta - \hat{\omega}_r}\hat{a}_{\omega_r}\right)\cdots\right)\right)\right)\right) \tag{8.9}$$

with the notation of (8.7). In view of the stability properties of $\hat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ (stability by convolution with another element of $\hat{H}\big(\mathcal{R}(\mathbb{Z})\big)$, a fortiori with an entire function, or by multiplication with a meromorphic function regular on $\mathbb{C} \setminus \mathbb{Z}^*$), this implies that *the functions* $\hat{\mathcal{V}}^{\boldsymbol{\omega}}$ *are resurgent*, as announced in the introduction to this section. We shall give more details on this later.

8.5. Here is a first consequence for the functions $\hat{\varphi}_n$ and $\hat{\psi}_n$:

Lemma 8.2. *For each* $n \in \mathbb{N}$,

$$\hat{\varphi}_n = \sum_{\|\boldsymbol{\omega}\|=n-1} \beta_{\boldsymbol{\omega}} \hat{\mathcal{V}}^{\boldsymbol{\omega}}, \quad \hat{\psi}_n = \sum_{\|\boldsymbol{\omega}\|=n-1} \beta_{\boldsymbol{\omega}} \hat{\mathcal{V}\!\!\!/}^{\boldsymbol{\omega}}, \tag{8.10}$$

with formally convergent series in $\mathbb{C}[[\zeta]]$, *and for each non-empty* $\boldsymbol{\omega}$ *such that* $\|\boldsymbol{\omega}\| = n-1$,

$$\beta_{\boldsymbol{\omega}} \hat{\mathcal{V}}^{\boldsymbol{\omega}} = \mathcal{S}_{\hat{\omega}_1}\mathcal{A}_{\omega_1}\cdots\mathcal{S}_{\hat{\omega}_r}\mathcal{A}_{\omega_r}\delta, \quad \beta_{\boldsymbol{\omega}} \hat{\mathcal{V}\!\!\!/}^{\boldsymbol{\omega}} = \tfrac{1}{\zeta-(n-1)}\mathcal{A}_{\omega_r}\mathfrak{S}_{\check{\omega}_{r-1}}\mathcal{A}_{\omega_{r-1}}\cdots\mathfrak{S}_{\check{\omega}_1}\mathcal{A}_{\omega_1}\delta, \tag{8.11}$$

with convolution operators

$$\mathcal{A}_\eta : \hat{\varphi} \mapsto \hat{a}_\eta * \hat{\varphi}, \quad \eta \in \Omega$$

and multiplication operators

$$\mathcal{S}_m : \hat{\varphi} \mapsto -\tfrac{n-m}{\zeta-m}\hat{\varphi}, \quad \mathfrak{S}_m : \hat{\varphi} \mapsto \tfrac{m+1}{\zeta-m}\hat{\varphi}, \quad m \in \mathbb{Z} \tag{8.12}$$

Proof. Formula (8.10) is a direct consequence of (8.6).

In order to deal with $\hat{\mathcal{V}}^{\boldsymbol{\omega}}$, we pass from $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ to $\widetilde{\boldsymbol{\omega}} = (\omega_r, \dots, \omega_1)$ and this exchanges $\hat{\omega}_i$ and $\check{\omega}_{r-i+1}$, thus (8.9) implies

$$\hat{\mathcal{V}}^{\boldsymbol{\omega}} = \frac{1}{\zeta - \check{\omega}_r}\Bigg(\hat{a}_{\omega_r} * \bigg(\frac{1}{\zeta - \check{\omega}_{r-1}}\Big(\hat{a}_{\omega_{r-1}} * \Big(\cdots\Big(\frac{1}{\zeta - \check{\omega}_1}\hat{a}_{\omega_1}\Big)\cdots\Big)\Big)\bigg)\Bigg).$$

Since $\check{\omega}_r = n - 1$, multiplying by $\beta_{\boldsymbol{\omega}} = (\check{\omega}_{r-1} + 1)\cdots(\check{\omega}_1 + 1)$, we get the second part of (8.11). The first part of this formula is obtained by multiplying (8.9) by $\beta_{\boldsymbol{\omega}}$ written in the form $\beta_{\boldsymbol{\omega}} = (n - \hat{\omega}_1)(n - \hat{\omega}_2)\cdots(n - \hat{\omega}_r)$ (indeed, $n - \hat{\omega}_1 = 1$ and $n - \hat{\omega}_i = \check{\omega}_{i-1} + 1$ for $2 \le i \le r$). □

8.6. The appearance of singularities in our problem is due to the multiplication operators $\mathcal{S}_{\hat{\omega}_i}$ or $\mathfrak{S}_{\check{\omega}_i}$. In view of Lemma 8.1 and formulas (8.11)–(8.12), we are led to introduce subspaces of $\hat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ formed of functions with smaller sets of singularities. We do this by considering Riemann surfaces $\mathcal{R}(\mathcal{P})$ slightly more general than $\mathcal{R}(\mathbb{Z})$.

Let $\mathcal{P}$ denote a subset of $\mathbb{Z}$. We define the Riemann surface $\mathcal{R}(\mathcal{P})$ as the set of all homotopy classes of rectifiable oriented paths which start from the origin and then avoid $\mathcal{P}$. The Riemann surface $\mathcal{R}(\mathcal{P})$ and the universal cover of $\mathbb{C} \setminus \mathcal{P}$ coincide if $0 \notin \mathcal{P}$; there is a difference between them when $0 \in \mathcal{P}$: there is no point which projects onto 0 in the second one, while the first one still has an "origin".

The space $\hat{H}\big(\mathcal{R}(\mathcal{P})\big)$ of all holomorphic functions of $\mathcal{R}(\mathcal{P})$ can be identified with the space of all $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ which admit an analytic continuation along any representative of any element of $\mathcal{R}(\mathcal{P})$. It can thus also be identified with the subspace of $\hat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ consisting of those functions holomorphic in $\mathcal{R}(\mathbb{Z})$, the branches of which are regular at each point of $\mathbb{Z} \setminus \mathcal{P}$.

We shall particularly be interested in two cases: $\mathcal{P}^- = n - \mathbb{N}^*$ and $\mathcal{P}^+ = \mathbb{N}$. Indeed, our aim is to show that the functions $\hat{\varphi}_n$ belong to $\hat{H}\big(\mathcal{R}(n - \mathbb{N}^*)\big)$ for any $n \in \mathbb{N}$ and that the functions $\hat{\psi}_n$ belong to $\hat{H}\big(\mathcal{R}(\mathbb{N})\big)$ for any $n \ge 1$, while $(\zeta + 1)\hat{\psi}_0(\zeta) \in \hat{H}\big(\mathcal{R}(\mathbb{N})\big)$.

One could prove (with the help of symmetrically contractile paths) that the spaces $\hat{H}\big(\mathcal{R}(\mathbb{N})\big)$, $\hat{H}\big(\mathcal{R}(-\mathbb{N}^*)\big)$ or $\hat{H}\big(\mathcal{R}(-\mathbb{N})\big)$ are stable by convolution because the corresponding sets $\mathcal{P}$ are stable by addition, but beware that this is not the case of $\hat{H}\big(\mathcal{R}(n - \mathbb{N}^*)\big)$ if $n \ge 2$.

8.7. As previously mentioned, for each $\boldsymbol{\omega} \ne \emptyset$, $\beta_{\boldsymbol{\omega}}\hat{\mathcal{V}}^{\boldsymbol{\omega}}$ and $\beta_{\boldsymbol{\omega}}\hat{\mathcal{V}}^{\boldsymbol{\omega}}$ belong to $\hat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ by virtue of general stability properties. But formula (8.11) permits a more elementary argument and more precise conclusions.

Indeed $\mathcal{A}_{\omega_r}\delta = \hat{a}_{\omega_r}$, resp. $\mathcal{A}_{\omega_1}\delta = \hat{a}_{\omega_1}$, is an entire function, which vanishes at the origin if $\omega_r = 0$, resp. $\omega_1 = 0$. Thus

$$\mathcal{S}_{\hat{\omega}_r}\mathcal{A}_{\omega_r}\delta = -\frac{n - \omega_r}{\zeta - \omega_r}\hat{a}_{\omega_r}, \quad \text{resp.} \quad \mathfrak{S}_{\check{\omega}_1}\mathcal{A}_{\omega_1}\delta = \frac{\omega_1 + 1}{\zeta - \omega_1}\hat{a}_{\omega_1}, \tag{8.13}$$

is meromorphic on $\mathbb{C}$ and regular at the origin if $\omega_r \neq 0$, resp. $\omega_1 \neq 0$, and entire if $\omega_r = 0$, resp. $\omega_1 = 0$. In fact,

$$\hat{\omega}_r \le n-1 \implies \mathcal{S}_{\hat{\omega}_r} \mathcal{A}_{\omega_r} \delta \in \hat{H}\big(\mathcal{R}(n-\mathbb{N}^*)\big), \quad \check{\omega}_1 \ge 0 \implies \mathcal{S}_{\check{\omega}_1} \mathcal{A}_{\omega_1} \delta \in \hat{H}\big(\mathcal{R}(\mathbb{N})\big).$$

Therefore, one can apply $r-1$ times the following

Lemma 8.3. *Suppose that $\mathcal{P} \subset \mathbb{Z}$, $\hat{\varphi} \in \hat{H}\big(\mathcal{R}(\mathcal{P})\big)$ and $\hat{b}$ is entire. Then $\hat{b} * \hat{\varphi} \in \hat{H}\big(\mathcal{R}(\mathcal{P})\big)$. If furthermore $\hat{s}$ is a meromorphic function, the poles of which all belong to $\mathcal{P}$ and with at most a simple pole at the origin, then $\hat{s}(\hat{b} * \hat{\varphi}) \in \hat{H}\big(\mathcal{R}(\mathcal{P})\big)$.*

Consider a rectifiable oriented path with arc-length parametrisation $\gamma\colon [0,T] \to \mathbb{C}$, such that $\gamma(0) = 0$ and $\gamma(t) \in \mathbb{C} \setminus \mathcal{P}$ for $0 < t \le T$. Denoting the analytic continuation of $\hat{\varphi}$ along γ by the same symbol $\hat{\varphi}$, we suppose moreover that

$$\big|\hat{\varphi}\big(\gamma(t)\big)\big| \le P(t)\,\mathrm{e}^{Ct}, \quad 0 \le t \le T,$$

with a continuous function P and a constant $C \ge 0$, and that there is a continuous monotonic non-decreasing function Q such that $|\hat{b}(\zeta)| \le Q\big(|\zeta|\big)\,\mathrm{e}^{C|\zeta|}$ for all $\zeta \in \mathbb{C}$. Then, for all $t \in [0,T]$, the analytic continuation of $\hat{b} * \hat{\varphi}$ at $\zeta = \gamma(t)$ satisfies

$$\hat{b} * \hat{\varphi}(\zeta) = \int_{\gamma_\zeta} \hat{b}(\zeta - \zeta')\hat{\varphi}(\zeta')\,\mathrm{d}\zeta', \quad \big|\hat{b} * \hat{\varphi}\big(\gamma(t)\big)\big| \le P * Q(t)\,\mathrm{e}^{Ct}, \tag{8.14}$$

with γ_ζ denoting the restriction $\gamma_{|[0,t]}$ and $P * Q(t) = \int_0^t P(t')Q(t-t')\,\mathrm{d}t'$.

Proof. The first statement and the first part of (8.14) are obtained by means of the Cauchy theorem: if γ_1 and γ_2 are two representatives of the same element of $\mathcal{R}(\mathcal{P})$ and $\xi = \pi([\gamma_1]) = \pi([\gamma_2])$, then $\int_{\gamma_1} \hat{b}(\xi - \xi')\hat{\varphi}(\xi')\,\mathrm{d}\xi'$ and $\int_{\gamma_2} \hat{b}(\xi - \xi')\hat{\varphi}(\xi')\,\mathrm{d}\xi'$ coincide; one can check that the function thus defined on $\mathcal{R}(\mathcal{P})$ is holomorphic and this is clearly an extension of $\hat{b} * \hat{\varphi}$. Moreover $\hat{b} * \hat{\varphi}$ vanishes at the origin, thus $\hat{s}(\hat{b} * \hat{\varphi}) \in \hat{H}\big(\mathcal{R}(\mathcal{P})\big)$ even if $\hat{s}$ has a simple pole at 0.

We thus have

$$\hat{b} * \hat{\varphi}\big(\gamma(t)\big) = \int_0^t \hat{b}\big(\gamma(t) - \gamma(t')\big)\hat{\varphi}\big(\gamma(t')\big)\dot{\gamma}(t')\,\mathrm{d}t'.$$

For almost every $t' \in [0,t]$, $|\dot{\gamma}(t')| = 1$ and $|\gamma(t) - \gamma(t')| \le t - t'$, whence $|\hat{b}\big(\gamma(t) - \gamma(t')\big)| \le Q(t-t')\,\mathrm{e}^{C(t-t')}$ by monotonicity of $\xi \mapsto Q(\xi)\,\mathrm{e}^{\xi}$. The conclusion follows. □

In view of Lemma 8.1 and formula (8.11), the first part of Lemma 8.3 implies

Corollary 8.1. *Let $n \in \mathbb{N}$ and $\boldsymbol{\omega}$ be a non-empty word such that $\|\boldsymbol{\omega}\| = n-1$. Then the function $\beta_{\boldsymbol{\omega}} \hat{\mathcal{V}}^{\boldsymbol{\omega}}$ belongs to $\hat{H}\big(\mathcal{R}(n - \mathbb{N}^*)\big)$ and the function*

$$\zeta \mapsto \big(\zeta - (n-1)\big)\beta_{\boldsymbol{\omega}} \hat{\mathcal{V}}^{\boldsymbol{\omega}}(\zeta)$$

belongs to $\hat{H}\big(\mathcal{R}(\mathbb{N})\big)$.

8.8. Our aim is now to exploit formula (8.11) and the quantitative information contained in Lemma 8.3 to produce upper bounds for

$$\big|\beta_\omega \widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\zeta)\big|, \quad \text{resp. } \big|\big(\zeta-(n-1)\big)\beta_\omega \widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\zeta)\big|$$

which will ensure the uniform convergence of the series (8.10) (up to the factor $\zeta-(n-1)$ for the second one) in any compact subset of $\mathcal{R}(n-\mathbb{N}^*)$, resp. $\mathcal{R}(\mathbb{N})$.

We first choose positive constants K, L, C such that

$$\big|\hat{a}_\eta(\zeta)\big| \le KL^\eta\,\mathrm{e}^{C|\zeta|}, \quad \zeta\in\mathbb{C},\ \eta\in\Omega. \tag{8.15}$$

This is possible, since $\sum \dfrac{a_\eta(x)}{x}y^{\eta+1} = \dfrac{A(x,y)-y}{y} \in \mathbb{C}\{x,y\}$ by assumption, thus one can find constants such that $\big|\frac{a_\eta(x)}{x}\big| \le KL^\eta$ for $|x|\le C^{-1}$ and use (8.1). We can also assume, possibly at the price of increasing of K, that

$$|\hat{a}_0(\zeta)| \le K|\zeta|\,\mathrm{e}^{C|\zeta|}, \quad \zeta\in\mathbb{C}, \tag{8.16}$$

since $a_0(x)\in x^2\mathbb{C}\{x\}$.

8.9. Next, we define exhaustions of $\mathcal{R}(n-\mathbb{N}^*)$, respectively of $\mathcal{R}(\mathbb{N})$, by subsets $\mathcal{R}_{\rho,N}(n-\mathbb{N}^*)$, respectively of $\mathcal{R}_{\rho,N}(\mathbb{N})$, in which we shall be able to derive appropriate bounds for our functions. Let $\rho\in\left]0,\frac{1}{2}\right[$ and $N\in\mathbb{N}^*$.

We denote by $\dot{\mathcal{R}}_{\rho,N}(n-\mathbb{N}^*)$ the subset of $\mathbb{C}$ obtained by removing the open discs $D(m,\rho)$ with radius ρ and integer centres $m\le n-1$, and removing also the points ζ such that the segment $[0,\zeta]$ intersect the open disc $D(-N,\rho)$ (i.e. the points which are hidden by $D(-N,\rho)$ to an observer located at the origin).

Similarly, we denote by $\dot{\mathcal{R}}_{\rho,N}(\mathbb{N})$ the subset of $\mathbb{C}$ obtained by removing the open discs $D(m,\rho)$ with radius ρ and integer centres $m\ge 0$, and removing also the points ζ such that the segment $[0,\zeta]$ intersect the open disc $D(N,\rho)$. Thus, with the notations $\mathcal{P}^- = n-\mathbb{N}^*$ and $\mathcal{P}^+=\mathbb{N}$,

$$\begin{aligned}\dot{\mathcal{R}}_{\rho,N}(\mathcal{P}^\pm) &= \{\,\zeta\in\mathbb{C} \mid \operatorname{dist}(\zeta,\mathcal{P}^\pm)\ge\rho \text{ and } \operatorname{dist}(\pm N,[0,\zeta])\ge\rho\,\}\\ &= \{\,\zeta\in\mathbb{C} \mid \operatorname{dist}\big(\zeta,\mathcal{P}^\pm\cup\pm\Sigma(\rho,N)\big)\ge\rho\,\},\end{aligned}$$

with the notation Σ introduced after the statement of Theorem 2; see Figure 1.

Now, for $\mathcal{P}=\mathcal{P}^\pm$, consider the rectifiable oriented paths γ which start at the origin and either stay in the disc $D(0,\rho)$, or leave it and then stay in $\dot{\mathcal{R}}_{\rho,N}(\mathcal{P})$. The homotopy classes of such paths form a set $\mathcal{R}_{\rho,N}(\mathcal{P})$ which we can identify with a subset of $\mathcal{R}(\mathcal{P})$.

Definition 8.2. If the arc-length parametrisation of a rectifiable oriented path $\gamma\colon[0,T]\to\mathbb{C}$ satisfies, for each $t\in[0,T]$,

$$\begin{aligned}0\le t\le\rho &\implies |\gamma(t)|=t,\\ t>\rho &\implies \gamma(t)\in\dot{\mathcal{R}}_{\rho,N}(\mathcal{P}),\end{aligned}$$

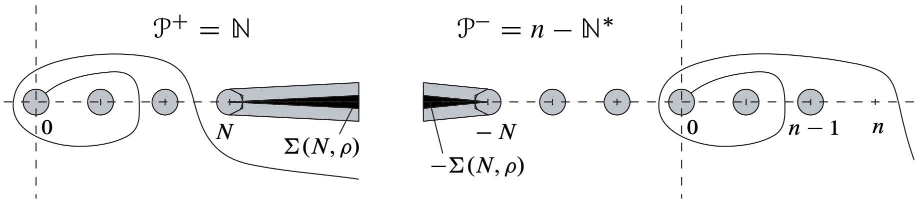

Figure 1. The set $\dot{\mathcal{R}}_{\rho,N}(\mathcal{P}^{\pm})$ and the image of a $(\rho, N, \mathcal{P}^{\pm})$-adapted path.

then we say that the parametrised path γ is $(\rho, N, \mathcal{P})$-adapted. We speak of infinite $(\rho, N, \mathcal{P})$-adapted path if γ is defined on $[0, +\infty[$.

One can characterize $\mathcal{R}_{\rho,N}(\mathcal{P})$ as follows: a point of $\mathcal{R}(\mathcal{P})$ belongs to $\mathcal{R}_{\rho,N}(\mathcal{P})$ iff it can be represented by a $(\rho, N, \mathcal{P})$-adapted path.

Observe that the projection onto $\mathbb{C}$ of $\mathcal{R}_{\rho,N}(\mathcal{P})$ is $\dot{\mathcal{R}}_{\rho,N}(\mathcal{P}) \cup D(0,\rho)$ (only for $\mathcal{P} = -\mathbb{N}^*$ is $D(0,\rho)$ contained in $\dot{\mathcal{R}}_{\rho,N}(\mathcal{P})$) and that $\mathcal{R}(\mathcal{P}) = \bigcup_{\rho,N} \mathcal{R}_{\rho,N}(\mathcal{P})$.

8.10. We now show how to control the operators $\mathcal{S}_m$ and $\mathfrak{S}_m$ uniformly in a set $\mathcal{R}_{\rho,N}(\mathcal{P}^{\pm})$:

Lemma 8.4. *Let $n \in \mathbb{N}$ and $\mathcal{S}_m, \mathfrak{S}_m$ as in (8.12), and consider the meromorphic functions $S_m = \mathcal{S}_m 1$ and $\mathfrak{S}_m = \mathfrak{S}_m 1$.*

Given ρ, N as above, there exist $\lambda > 0$ which depends only on ρ, N and $\lambda_n > 0$ which depends only on ρ, N, n such that, for $m \in \mathcal{P} \setminus \{0\}$,

$$\text{if } \mathcal{P} = n - \mathbb{N}^*: \quad |S_m(\zeta)| \le \lambda_n \quad \text{for } \zeta \in \dot{\mathcal{R}}_{\rho,N}(\mathcal{P}) \cup D(0,\rho) \tag{8.17}$$

$$\text{if } \mathcal{P} = \mathbb{N}: \quad |\mathfrak{S}_m(\zeta)| \le \lambda \quad \text{for } \zeta \in \dot{\mathcal{R}}_{\rho,N}(\mathcal{P}) \cup D(0,\rho) \tag{8.17$'$}$$

and

$$|S_0(\zeta)| \le \lambda_n, \quad |\mathfrak{S}_0(\zeta)| \le \lambda, \quad \text{for } \zeta \in \mathbb{C} \setminus D(0,\rho) \tag{8.18}$$

$$|S_0(\zeta)| \le \frac{\rho\lambda_n}{|\zeta|}, \quad |\mathfrak{S}_0(\zeta)| \le \frac{\rho\lambda}{|\zeta|}, \quad \text{for } \zeta \in D(0,\rho). \tag{8.19}$$

One can take $\lambda = (N+1)\rho^{-1}$ and $\lambda_n = (|n| + N)\rho^{-1}$.

Proof. Let $m \in \mathcal{P} \setminus \{0\}$ and $\zeta \in \dot{\mathcal{R}}_{\rho,N}(\mathcal{P}) \cup D(0,\rho)$, thus $|\zeta - m| \ge \rho$.

Consider first the case $\mathcal{P} = \mathbb{N}$. If $m \ge N$, then $|\zeta - m| \ge \frac{\rho|m|}{N}$ by Thales theorem; thus $\frac{1}{|\zeta - m|} \le \rho^{-1}$ and $\left|\frac{m}{\zeta - m}\right| \le N\rho^{-1}$ for any $m \in \mathbb{N}^*$. Therefore $|\mathfrak{S}_m(\zeta)| = \left|\frac{m+1}{\zeta - m}\right| \le \lambda = (N+1)\rho^{-1}$. Since $\lambda \ge \rho^{-1}$, $\mathfrak{S}_0(\zeta) = 1/\zeta$ also satisfies the required inequalities.

When $\mathcal{P} = n - \mathbb{N}^*$, one argues similarly except that the case $N \le m \le n-1$ must be treated separately. $\square$

8.11. Combining the previous two lemmas, we get

Lemma 8.5. *Let us fix n, ρ, N as above, K, L, C as in (8.15)–(8.16) and λ, λ_n as in Lemma 8.4. Suppose that $\mathcal{P} = n - \mathbb{N}^*$ or $\mathbb{N}$, $\gamma\colon [0,T] \to \mathbb{C}$ is $(\rho, N, \mathcal{P})$-adapted and $\widehat{\varphi} \in \widehat{H}\big(\mathcal{R}(\mathcal{P})\big)$ satisfies*

$$\big|\widehat{\varphi}\big(\gamma(t)\big)\big| \le P(t)\,\mathrm{e}^{Ct}, \quad 0 \le t \le T,$$

with a continuous monotonic non-decreasing function P and a constant $C \ge 0$. Assume $m \in \mathcal{P}$, with the restriction $m \ne 0$ if $n = 0$ and $\mathcal{P} = -\mathbb{N}^*$.

Then, for any $\eta \in \Omega$,

$$\mathcal{P} = n - \mathbb{N}^* \implies \mathcal{S}_m \mathcal{A}_\eta \widehat{\varphi} \in \widehat{H}\big(\mathcal{R}(n - \mathbb{N}^*)\big), \quad \mathcal{P} = \mathbb{N} \implies \mathfrak{S}_m \mathcal{A}_\eta \widehat{\varphi} \in \widehat{H}\big(\mathcal{R}(\mathbb{N})\big),$$

and, in the first case,

$$m \ne 0 \ \text{or}\ \eta = 0 \implies \big|\mathcal{S}_m \mathcal{A}_\eta \widehat{\varphi}\big(\gamma(t)\big)\big| \le \lambda_n K L^\eta (1 * P)(t)\,\mathrm{e}^{Ct} \tag{8.20}$$

$$m = 0 \ \text{and}\ \eta \ne 0 \implies \big|\mathcal{S}_m \mathcal{A}_\eta \widehat{\varphi}\big(\gamma(t)\big)\big| \le \lambda_n K L^\eta \big((\delta + 1) * P\big)(t)\,\mathrm{e}^{Ct} \tag{8.21}$$

for all $t \in [0,T]$, while in the second case the function $\mathfrak{S}_m \mathcal{A}_\eta \widehat{\varphi}$ satisfies the same inequalities with λ replacing λ_n.

Proof. We suppose $\mathcal{P} = n - \mathbb{N}^*$ and show the properties for $\mathcal{S}_m \mathcal{A}_\eta \widehat{\varphi}$ only, the other case being similar. Since $\mathcal{S}_m \mathcal{A}_\eta \widehat{\varphi} = S_m(\widehat{a}_\eta * \widehat{\varphi})$, this function belongs to $\widehat{H}\big(\mathcal{R}(\mathcal{P})\big)$ by the first part of Lemma 8.3. In view of (8.15)–(8.16), the second part of this lemma yields

$$\big|\mathcal{A}_\eta \widehat{\varphi}\big(\gamma(t)\big)\big| \le K L^\eta (1 * P)(t)\,\mathrm{e}^{Ct} \tag{8.22}$$

$$\big|\mathcal{A}_0 \widehat{\varphi}\big(\gamma(t)\big)\big| \le K (I * P)(t)\,\mathrm{e}^{Ct} \tag{8.23}$$

for all $t \in [0,T]$, with $I(t) \equiv t$ (notice that the first inequality holds if $\eta = 0$ as well).

If $m \ne 0$, then (8.17) yields the desired inequality for $\big|\mathcal{S}_m \mathcal{A}_\eta \widehat{\varphi}\big(\gamma(t)\big)\big|$.

Suppose $m = 0$; thus $n \ne 0$ by assumption. We observe that, if $t > \rho$, then $\gamma(t) \in \dot{\mathcal{R}}_{\rho,N}(\mathcal{P})$ has modulus $> \rho$ and (8.18) yields $\big|S_0\big(\gamma(t)\big)\big| \le \lambda_n$, whereas if $t \le \rho$, then $|\gamma(t)| = |t|$ and (8.19) yields $\big|S_0\big(\gamma(t)\big)\big| \le \frac{\rho\lambda_n}{t}$.

Thus, if $m = 0$ and $\eta = 0$, then (8.22) yields the desired inequality when $t > \rho$ and (8.23) yields $\big|\mathcal{S}_0 \mathcal{A}_0 \widehat{\varphi}\big(\gamma(t)\big)\big| \le K\rho\lambda_n \frac{I * P(t)}{t}\,\mathrm{e}^{Ct}$ for $t \le \rho$, which is sufficient since $\frac{I * P(t)}{t} = \frac{1}{t}\int_0^t t' P(t - t')\,\mathrm{d}t' \le 1 * P(t)$ and $\rho < 1$.

We conclude with the case where $m = 0$ and $\eta \ne 0$. Using (8.22), we obtain the result when $t > \rho$, since $1 * P \le P + 1 * P$. When $t \le \rho$, we get $\big|\mathcal{S}_0 \mathcal{A}_\eta \widehat{\varphi}\big(\gamma(t)\big)\big| \le K L^\eta \rho\lambda_n \frac{1 * P(t)}{t}\,\mathrm{e}^{Ct}$, which is sufficient since $\frac{1 * P(t)}{t} = \frac{1}{t}\int_0^t P(t')\,\mathrm{d}t' \le P(t)$. $\square$

8.12. *End of the proof of Theorem 2*: case of $\widehat{\varphi}_n$.

Let $n \in \mathbb{N}$. According to (8.10), the formal series $\widehat{\varphi}_n$ can be written as the formally convergent series $\sum_{\|\boldsymbol{\omega}\|=n-1} \beta_{\boldsymbol{\omega}} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}$. Let $\mathcal{P} = n - \mathbb{N}^*$; according to Corollary 8.1 each $\beta_{\boldsymbol{\omega}} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}$ converges to a function of $\widehat{H}\big(\mathcal{R}(\mathcal{P})\big)$, it is thus sufficient to check the uniform convergence of the above series *as a series of holomorphic functions* in each compact subset of $\mathcal{R}(\mathcal{P})$ and to give appropriate bounds. Let us fix $\rho \in \left]0, \frac{1}{2}\right[$, $N \in \mathbb{N}^*$ and $K, L, C, \lambda, \lambda_n$ as in Lemma 8.5.

– We first show that, for any $(\rho, N, \mathcal{P})$-adapted path γ (infinite or not) and for any $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r) \in \Omega^r$ with $r \geq 1$ and $\|\boldsymbol{\omega}\| = n - 1$, one has for all t

$$\big|\beta_{\boldsymbol{\omega}} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}\big(\gamma(t)\big)\big| \leq (\lambda_n K)^r L^{n-1} \widehat{P}_r(t)\, \mathrm{e}^{Ct}, \quad \widehat{P}_r = (\delta + 1)^{*\lfloor r/2 \rfloor} * 1^{*\lceil r/2 \rceil}, \tag{8.24}$$

with the same notation as in (3.8) for $\lceil r/2 \rceil$ and $\lfloor r/2 \rfloor = r - \lceil r/2 \rceil$. Observe that $\widehat{P}_r$ is a polynomial with non-negative coefficients.

If $\omega_r = \check{\omega}_r \neq 0$, then (8.15) and (8.17) yield $|\mathcal{S}_{\hat{\omega}_r} \hat{a}_{\omega_r}(\zeta)| \leq \lambda_n K L^{\omega_r} \mathrm{e}^{C|\zeta|}$ for all $\zeta \in \mathcal{R}(\mathcal{P})$. The same inequality holds also if $\omega_r = 0$ (use (8.15) and (8.18) if $|\zeta| > \rho$, and (8.16) and (8.19) if $|\zeta| \leq \rho$). Therefore

$$\big|\mathcal{S}_{\hat{\omega}_r} \hat{a}_{\omega_r}\big(\gamma(t)\big)\big| \leq \lambda_n K L^{\omega_r}\, \mathrm{e}^{Ct}, \quad t \geq 0 \tag{8.25}$$

(since $|\gamma(t)| \leq t$). Since Lemma 8.1 implies $\hat{\omega}_1, \dots, \hat{\omega}_{r-1} \leq n - 1$, we can apply $r - 1$ times Lemma 8.5 and get

$$\big|\beta_{\boldsymbol{\omega}} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}\big(\gamma(t)\big)\big| \leq (\lambda_n K)^r L^{n-1} \big((\delta + 1)^{*(r-a)} * 1^{*a}\big)(t)\, \mathrm{e}^{Ct}, \tag{8.26}$$

with $a = \operatorname{card}\{\, i \in [1, r] \mid \hat{\omega}_i \neq 0 \text{ or } \omega_i = 0 \,\}$. But $a \geq \lceil r/2 \rceil$, as was shown in Lemma 3.2, hence the polynomial expression in t appearing in the right-hand side of (8.26) can be written $(\delta+1)^{*(r-a)} * 1^{*(a - \lceil r/2 \rceil)} * 1^{*\lceil r/2 \rceil} \leq (\delta+1)^{*(r - \lceil r/2 \rceil)} * 1^{*\lceil r/2 \rceil}$, which yields (8.24).

– We have

$$\begin{aligned} \operatorname{card}\{\, \boldsymbol{\omega} \in \Omega^r \mid \|\boldsymbol{\omega}\| = n - 1 \,\} &= \operatorname{card}\{\, k \in \mathbb{N}^r \mid \|k\| = n + r - 1 \,\} \\ &= \binom{n + 2(r-1)}{r - 1} \leq 2^{n + 2(r-1)}, \end{aligned}$$

hence, for each $r \geq 1$,

$$\sum_{r(\boldsymbol{\omega}) = r, \|\boldsymbol{\omega}\| = n-1} \big|\beta_{\boldsymbol{\omega}} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}\big(\gamma(t)\big)\big| \leq 2\lambda_n K (2L)^{n-1} \Lambda_n^{r-1} \widehat{P}_r(t)\, \mathrm{e}^{Ct} \tag{8.27}$$

with $\Lambda_n = 4\lambda_n K$. But $\widetilde{P}_r(z) = \mathcal{B}^{-1} \widehat{P}_r = (1 + z^{-1})^{\lfloor r/2 \rfloor} z^{-\lceil r/2 \rceil}$ gives rise to

$$\widetilde{\Phi}_n(z) = \sum_{r \geq 1} \Lambda_n^{r-1} \widetilde{P}_r(z) = z^{-1}\big(1 + \Lambda_n(1 + z^{-1})\big)\big(1 - \Lambda_n^2(z^{-1} + z^{-2})\big)^{-1}$$

which is convergent (with non-negative coefficients), so $\sum_{r\ge1}\Lambda_n^{r-1}\widehat{P}_r(t)=\mathcal{B}\widetilde{\Phi}_n(t)$ is convergent for all t. Therefore $\widehat{\varphi}_n$ is the sum of a series of holomorphic functions uniformly convergent in every compact subset of $\mathcal{R}_{\rho,N}(\mathcal{P})$ satisfying

$$\left|\widehat{\varphi}_n\big(\gamma(t)\big)\right|\le 2\lambda_n K(2L)^{n-1}\mathcal{B}\widetilde{\Phi}_n(t)\,\mathrm{e}^{Ct}.$$

– We conclude by using inequalities of the form (8.1) to bound $\mathcal{B}\widetilde{\Phi}_n$: one can check that $|z|\ge 4\Lambda_n^2$ implies $|z\widetilde{\Phi}_n(z)|\le 2(2+\Lambda_n)$, hence

$$\mathcal{B}\widetilde{\Phi}_n(t)\le 2(2+\Lambda_n)\,\mathrm{e}^{4\Lambda_n^2 t}.$$

In view of the explicit dependence of λ_n on n indicated in Lemma 8.4, we easily get inequalities of the form (8.3) (possibly with larger constants K, L, C).

8.13. *End of the proof of Theorem* 2: case of $\widehat{\psi}_n$.

We only indicate the inequalities that one obtains when adapting the previous arguments to the case of $\widehat{\psi}_n$. Let $\mathcal{P}=\mathbb{N}$ and

$$\widehat{\chi}_n(\zeta)=\big(\zeta-(n-1)\big)\widehat{\psi}_n(\zeta),\qquad \widehat{\mathcal{W}}^{\boldsymbol{\omega}}(\zeta)=\big(\zeta-(n-1)\big)\beta_{\boldsymbol{\omega}}\widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\zeta).$$

The initial bound corresponding to (8.25) is $\left|\mathcal{S}_{\check{\omega}_1}\hat{a}_{\omega_1}\big(\gamma(t)\big)\right|\le\lambda K L^{\omega_1}\,\mathrm{e}^{Ct}$. This yields

$$\left|\widehat{\mathcal{W}}^{\boldsymbol{\omega}}\big(\gamma(t)\big)\right|\le K(\lambda K)^{r-1}L^{n-1}\widehat{Q}_r(t)\,\mathrm{e}^{Ct},\qquad \widehat{Q}_r=(\delta+1)^{*\lceil\frac{r}{2}-1\rceil}*1^{*\lfloor\frac{r}{2}+1\rfloor}\tag{8.28}$$

after $r-2$ applications of Lemma 8.5 (with an intermediary inequality analogous to (8.26) but involving $b=1+\operatorname{card}\{\,i\in[1,r-1]\mid\check{\omega}_i\ne0\text{ or }\omega_i=0\,\}\ge\lfloor\frac{r}{2}+1\rfloor$ instead of a).

Therefore $\left|\widehat{\chi}_n\big(\gamma(t)\big)\right|\le 2K(2L)^{n-1}\mathcal{B}\widetilde{\Psi}(t)\,\mathrm{e}^{Ct}$, with

$$\widetilde{\Psi}(z)=\sum_{r\ge1}\Lambda^{r-1}\mathcal{B}^{-1}\widehat{Q}_r(z)=z^{-1}(1+\Lambda z^{-1})\big(1-\Lambda^2(z^{-1}+z^{-2})\big)^{-1},\quad \Lambda=4\lambda K,$$

whence $\left|\widehat{\chi}_n\big(\gamma(t)\big)\right|\le K_1(2L)^n\,\mathrm{e}^{C_1t}$, with suitable constants K_1, C_1 independent of n.

This is the desired conclusion when $n=0$. When $n\ge1$, we can pass from $\widehat{\chi}_n$ to $\widehat{\psi}_n$ since $|\gamma(t)-(n-1)|\ge\rho$, with only one exception; namely, if $n=1$ and $t<\rho$, then we only have a bound for $|\zeta\widehat{\psi}_1(\zeta)|$ with $\zeta=\gamma(t)\in D(0,\rho)$, but in that case the analyticity of $\widehat{\chi}_1$ at the origin of $\mathcal{R}(\mathbb{N})$ is sufficient since we know that its Taylor series has no constant term.

8.14. *Proof of inequalities* (8.5). They will follow from a lemma which has its own interest.

Lemma 8.6. *For every $n\in\mathbb{N}$, the following identity holds in $\mathbb{C}[[x]]$:*

$$\varphi_n=\sum_{\substack{s\ge1,\ n_1,\dots,n_s\ge0\\ n_1+\dots+n_s=n+s-1}}\frac{(-1)^s}{s}\binom{n+s-1}{s-1}\psi_{n_1}\cdots\psi_{n_s},\tag{8.29}$$

where the right-hand side is a formally convergent series.

Proof. This is the consequence of the following version of Lagrange inversion formula: If $\chi(t,y) \in \mathbb{C}[[t,y]]$, then the formal transformation

$$(t,x,y) \mapsto \big(t,x,y - x\chi(t,y)\big)$$

has an inverse of the form $(t,x,y) \mapsto \big(t,x,\mathcal{Y}(t,x,y)\big)$ with $\mathcal{Y} \in \mathbb{C}[[t,x,y]]$ given by

$$\mathcal{Y}(t,x,y) = y + \sum_{s\geq 1} \frac{x^s}{s!} \Big(\frac{\partial}{\partial y}\Big)^{s-1} \big(\chi(t,y)^s\big). \tag{8.30}$$

(Proof: The transformation is invertible, because its 1-jet is, and the inverse must be of the form $\big(t,x,\mathcal{Y}(t,x,y)\big)$ with $\mathcal{Y}(t,0,y) = y$ and $\partial_y \mathcal{Y}(t,0,y) = 1$. It is thus sufficient to check the formula

$$\partial_x^s \mathcal{Y}(t,x,y) = \Big(\frac{\partial}{\partial y}\Big)^{s-1} \Big[\Big(\chi\big(t,\mathcal{Y}(t,x,y)\big)\Big)^s \partial_y \mathcal{Y}(t,x,y)\Big], \quad s \geq 1$$

by induction on s, which is easy.)

Since $\psi_n(x) \in x\mathbb{C}[[x]]$, we can apply this with $\chi(t,y) = \sum_{n\geq 0} \chi_n(t) y^n$ where $\chi_n(x) = -\frac{\psi_n(x)}{x}$: this way $y - x\chi(x,y) = y + \sum_{n\geq 0} \psi_n(x) y^n = \psi(x,y)$, and (8.30) yields

$$\varphi(x,y) = y + \sum_{n\geq 0} \varphi_n(x) y^n = \sum_{s\geq 1} \frac{(-1)^s}{s!} \Big(\frac{\partial}{\partial y}\Big)^{s-1} \Big[\Big(\sum_{n\geq 0} \psi_n(x) y^n\Big)^s\Big]$$

by specialization to $t = x$, whence the result follows (one gets a formally convergent series because $\psi_n(x) \in x\mathbb{C}[[x]]$). $\square$

As a consequence, we get

$$\widehat{\varphi}_n = \sum_{\substack{s\geq 1,\ n_1,\dots,n_s\geq 0\\ n_1+\dots+n_s=n+s-1}} \frac{(-1)^s}{s} \binom{n+s-1}{s-1} \widehat{\psi}_{n_1} * \dots * \widehat{\psi}_{n_s}$$

a priori in $\mathbb{C}[[\zeta]]$, but the right-hand side is also a series of holomorphic functions and inequalities (8.4) will yield uniform convergence in every compact subset of the principal sheet of $\mathcal{R}(\mathbb{Z})$.

Indeed, let $\rho \in \left]0, \frac{1}{2}\right[$. The domain considered in (8.5) consists of those $\zeta \in \mathbb{C}$ such that the segment $[0,\zeta]$ does not meet the open discs $D(-1,\rho)$ and $D(1,\rho)$. All the $\widehat{\psi}_n$'s are holomorphic in this domain $\mathcal{D}_\rho$ (we had to delete the disc around -1 only because of $\widehat{\psi}_0$).

Since $\mathcal{D}_\rho$ is star-shaped with respect to 0, the analytic continuation of the convolution product of any two functions $\widehat{\varphi}$ and $\widehat{\psi}$ holomorphic in $\mathcal{D}_\rho$ is defined by formula (8.2) regardless of the size of $|\zeta|$. If moreover one has inequalities of the form $|\widehat{\varphi}(\zeta)| \leq \Phi(|\zeta|)\, \mathrm{e}^{C|\zeta|}$ and $|\widehat{\psi}(\zeta)| \leq \Psi(|\zeta|)\, \mathrm{e}^{C|\zeta|}$ in $\mathcal{D}_\rho$, then the inequality

$|\widehat{\varphi} * \widehat{\psi}(\zeta)| \le \Phi * \Psi(|\zeta|)\, e^{C|\zeta|}$ holds in $\mathcal{D}_\rho$. Hence

$$|\widehat{\varphi}_n(\zeta)| \le \sum_{\substack{s\ge1,\ n_1,\dots,n_s\ge0\\ n_1+\dots+n_s=n+s-1}} \frac{1}{s}\binom{n+s-1}{s-1} K^s L^{n+s-1} M_s(|\zeta|)\, e^{C|\zeta|}, \quad \zeta\in\mathcal{D}_\rho,$$

with $M_s(\zeta) = 1^{*s}(\zeta) = \frac{\zeta^{s-1}}{(s-1)!}$. The conclusion follows since the right-hand side is less than $K(4L)^n\, e^{(C+8KL)|\zeta|}$.

9 The $\widetilde{\mathcal{V}}^{\omega}$'s as resurgence monomials – introduction to alien calculus

9.1. Resurgence theory means much more than Borel–Laplace summation. It incorporates a study of the role of the singularities which appear in the Borel plane (i.e. the plane of the complex variable ζ), which can be performed through the so-called *alien calculus*.

We shall now recall Écalle's definitions in a particular case which will suffice for the saddle-node problem. We shall give less details than in the previous section; see e.g. [14], §2.3 for more information (and *op. cit.*, §3 for an outline of the general case and more references).

The reader will thus find in this section the definition of a subalgebra $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ of $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$, which is called the algebra of *simple resurgent functions over* $\mathbb{Z}$, and of a collection of operators Δ_m, $m\in\mathbb{Z}^*$, which are derivations of $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ called *alien derivations*. Alien calculus consists in the proper use of these derivations.

We shall see that the formal series $\widetilde{\mathcal{V}}^{\omega_1,\dots,\omega_r}$ belong to $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ and study the effect of the alien derivations on them.

9.2. Let $\widehat{\varphi}$ be holomorphic in an open subset U of $\mathbb{C}$ and $\omega\in\partial U$. We say that $\widehat{\varphi}$ has a *simple singularity* at ω if there exist $C\in\mathbb{C}$ and $\widehat{\chi}(\zeta), \mathrm{reg}(\zeta)\in\mathbb{C}\{\zeta\}$ such that

$$\widehat{\varphi}(\zeta) = \frac{C}{2\pi\mathrm{i}(\zeta-\omega)} + \frac{1}{2\pi\mathrm{i}}\widehat{\chi}(\zeta-\omega)\log(\zeta-\omega) + \mathrm{reg}(\zeta-\omega) \tag{9.1}$$

for ζ close enough to ω. The *residuum* C and the *variation* $\widehat{\chi}$ are then determined by $\widehat{\varphi}$ (independently of the choice of the branch of the logarithm):

$$C = 2\pi\mathrm{i}\lim_{\substack{\zeta\to\omega\\ \zeta\in U}}(\zeta-\omega)\widehat{\varphi}(\zeta), \quad \widehat{\chi}(\zeta) = \widehat{\varphi}(\omega+\zeta) - \widehat{\varphi}(\omega+\zeta\, e^{-2\pi\mathrm{i}}),$$

where it is understood that considering $\omega+\zeta\, e^{-2\pi\mathrm{i}}$ means following the analytic continuation of $\widehat{\varphi}$ along the circular path $t\in[0,1]\mapsto\omega+\zeta\, e^{-2\pi\mathrm{i}t}$ (which is possible when starting from $\omega+\zeta\in U$ provided $|\zeta|$ is small enough). In this situation let us use the notation

$$\operatorname{sing}_\omega \widehat{\varphi} = C\,\delta + \widehat{\chi} \in \mathbb{C}\,\delta \oplus \mathbb{C}\{\zeta\}.$$

We recall that $\widehat{\boldsymbol{R}}_{\mathbb{Z}} = \mathbb{C}\,\delta \oplus \widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$.

Definition 9.1. A simple resurgent function over $\mathbb{Z}$ is any $c\,\delta + \widehat{\varphi} \in \widehat{\boldsymbol{R}}_{\mathbb{Z}}$ such that all branches of the holomorphic function $\widehat{\varphi} \in \widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ only have simple singularities (necessarily located at points of $\mathbb{Z}$). The space of simple resurgent functions over $\mathbb{Z}$ will be denoted $\widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$.

It turns out that $\widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$ *is stable by convolution:* it is a subalgebra of $\widehat{\boldsymbol{R}}_{\mathbb{Z}}$. This is the *convolutive model of the algebra of simple resurgent functions*. The *formal model* is defined as $\widetilde{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}} = \mathcal{B}^{-1}(\widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}})$, which is a subalgebra of $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$.

9.3. For a simple resurgent function $c\,\delta + \widehat{\varphi}$ and a path γ which starts from 0 and then avoids $\mathbb{Z}$, we shall denote by $\operatorname{cont}_\gamma \widehat{\varphi}$ the analytic continuation of $\widehat{\varphi}$ along γ: this function is analytic in a neighbourhood of the endpoint of γ and admits itself an analytic continuation along all the paths which avoid $\mathbb{Z}$. If the endpoint of γ is close to m (say at a distance $< \frac{1}{2}$), then the singularity $\operatorname{sing}_m(\operatorname{cont}_\gamma \widehat{\varphi}) \in \mathbb{C}\,\delta \oplus \mathbb{C}\{\zeta\}$ is well-defined (notice that it depends on the branch under consideration, i.e. on γ, and not only on m and $\widehat{\varphi}$). It is easy to see that $\operatorname{sing}_m(\operatorname{cont}_\gamma \widehat{\varphi})$ is itself a simple resurgent function; we thus have, for γ and m as above, a $\mathbb{C}$-linear operator $c\,\delta + \widehat{\varphi} \mapsto \operatorname{sing}_m(\operatorname{cont}_\gamma \widehat{\varphi})$ from $\widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$ to itself.

Definition 9.2. Let $m \in \mathbb{Z}^*$. If $m \geq 1$, we define an operator from $\widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$ to itself by using 2^{m-1} particular paths γ:

$$\Delta_m(c\,\delta + \widehat{\varphi}) = \sum_{\varepsilon \in \{+,-\}^{m-1}} \frac{p_\varepsilon! q_\varepsilon!}{m!} \operatorname{sing}_m(\operatorname{cont}_{\gamma_\varepsilon} \widehat{\varphi}) \tag{9.2}$$

where p_ε and $q_\varepsilon = m - 1 - p_\varepsilon$ denote the numbers of signs '$+$' and of signs '$-$' in the sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{m-1})$, and the oriented path γ_ε connects 0 and m following the segment $]0, m[$ but circumventing the intermediary integer points k to the right if $\varepsilon_k = +$ and to the left if $\varepsilon_k = -$.

If $m \leq -1$, the $\mathbb{C}$-linear operator Δ_m is defined similarly, using the $2^{|m|-1}$ paths γ_ε which follow $]0, -|m|[$ but circumvent the intermediary integer points $-k$ to the right if $\varepsilon_k = +$ and to the left if $\varepsilon_k = -$.

Proposition 9.1. *For each $m \in \mathbb{Z}^*$, the operator Δ_m is a $\mathbb{C}$-linear derivation of $\widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$.*

(For the proof, see [3] or [14], §2.3; see also Lemma 9.2 and the comment on it below.)

By conjugacy by the formal Borel transform $\mathcal{B}$, we get a derivation of $\widetilde{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$, still denoted Δ_m since there is no risk of confusion. The operator Δ_m is called the *alien derivation of index m* (either in the convolutive model $\widehat{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ or in the formal model $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$).

One can easily check from Definition 9.2 that

$$[\partial, \Delta_m] = m\Delta_m \ \text{ in } \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}, \quad [\widehat{\partial}, \Delta_m] = m\Delta_m \ \text{ in } \widehat{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}, \tag{9.3}$$

where ∂ denotes the natural derivation $\frac{\mathrm{d}}{\mathrm{d}z}$ of $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ and $\widehat{\partial}$ is the corresponding derivation $c\,\delta + \widehat{\varphi}(\zeta) \mapsto -\zeta\widehat{\varphi}(\zeta)$ of $\widehat{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$.

9.4. We shall see that the operators Δ_m are independent in a strong sense (see Theorem 8 below). This will rely on a study of the way the alien derivations act on the resurgent functions $\widehat{\mathcal{V}}^{\omega_1,\dots,\omega_r}$.

In this article, for the sake of simplicity, we shall not introduce the larger commutative algebras $\widehat{\mathrm{RES}}^{\mathrm{simp}}$ and $\widetilde{\mathrm{RES}}^{\mathrm{simp}}$ of simple resurgent functions "over $\mathbb{C}$", i.e. with simple singularities in the Borel plane which can be located anywhere. In these algebras act alien derivations indexed by any non-zero complex number. One could easily adapt the arguments that we are about to develop to the study of the alien derivations Δ_ω, $\omega \in \mathbb{C}^*$, in $\widetilde{\mathrm{RES}}^{\mathrm{simp}}$.

One can also define an even larger commutative algebra of resurgent functions, without any restriction on the nature of the singularities to be encountered in the Borel plane, on which act alien derivations Δ_ω indexed by points ω of the Riemann surface of the logarithm, but there is no formal counterpart contained in $\mathbb{C}[[z^{-1}]]$ (see e.g. [14], §3, and the references therein).

9.5. We now check that the formal series $\widetilde{\mathcal{V}}^{\omega_1,\dots,\omega_r}$ are simple resurgent functions and slightly extend at the same time their definition.

Lemma 9.1. *Let $A = \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ and $\Omega \subset \mathbb{Z}$. Assume that $a = (\hat{a}_\eta)_{\eta\in\Omega}$ is a family of entire functions; if $0 \in \Omega$, we assume furthermore that $\hat{a}_0(0) = 0$. Let $\widetilde{a}_\eta = \mathcal{B}^{-1}\hat{a}_\eta \in z^{-1}\mathbb{C}[[z^{-1}]]$.*

Then the equations $\widetilde{\mathcal{V}}^{\emptyset}_a = \widetilde{\mathcal{V}\!\!\!/}^{\emptyset}_a = 1$ and, for $\boldsymbol{\omega} \in \Omega^\bullet$ non-empty,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + \|\boldsymbol{\omega}\|\right)\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}_a = \widetilde{a}_{\omega_1}\widetilde{\mathcal{V}}^{`\boldsymbol{\omega}}_a, \quad \left(\frac{\mathrm{d}}{\mathrm{d}z} + \|\boldsymbol{\omega}\|\right)\widetilde{\mathcal{V}\!\!\!/}^{\boldsymbol{\omega}}_a = -\widetilde{a}_{\omega_r}\widetilde{\mathcal{V}\!\!\!/}^{\boldsymbol{\omega}'}_a \tag{9.4}$$

(with $`\boldsymbol{\omega}$ denoting $\boldsymbol{\omega}$ deprived from its first letter, $\boldsymbol{\omega}'$ denoting $\boldsymbol{\omega}$ deprived from its last letter and $\|\boldsymbol{\omega}\|$ the sum of the letters of $\boldsymbol{\omega}$) determine inductively two moulds $\widetilde{\mathcal{V}}^\bullet_a, \widetilde{\mathcal{V}\!\!\!/}^\bullet_a \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$, which are symmetral and mutually inverse for mould multiplication.

Proof. A mere adaptation of Lemma 3.2 and Proposition 5.5 (in which the fact that $\Omega = \mathcal{N}$ played no role) shows that $\widetilde{\mathcal{V}}^\bullet_a$ and $\widetilde{\mathcal{V}\!\!\!/}^\bullet_a$ are well-defined by (9.4) as moulds on Ω *with values in* $\mathbb{C}[[z^{-1}]]$, with $\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}_a, \widetilde{\mathcal{V}\!\!\!/}^{\boldsymbol{\omega}}_a \in z^{-1}\mathbb{C}[[z^{-1}]]$ as soon as $\boldsymbol{\omega} \neq \emptyset$, that they are related by the involution S of Proposition 5.2:

$$\widetilde{\mathcal{V}\!\!\!/}^\bullet_a = S\widetilde{\mathcal{V}}^\bullet_a \tag{9.5}$$

and symmetral, hence mutually inverse.

The formal Borel transforms are given by $\widehat{\mathcal{V}}_a^{\emptyset} = \widehat{\mathcal{V}}_a^{\emptyset} = \delta$ and, for $\boldsymbol{\omega} \neq \emptyset$,

$$\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}(\zeta) = -\frac{1}{\zeta - \|\boldsymbol{\omega}\|}\big(\hat{a}_{\omega_1} * \widehat{\mathcal{V}}_a^{`\boldsymbol{\omega}}\big), \quad \widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}(\zeta) = \frac{1}{\zeta - \|\boldsymbol{\omega}\|}\big(\hat{a}_{\omega_r} * \widehat{\mathcal{V}}_a^{\boldsymbol{\omega}`}\big), \tag{9.6}$$

where the right-hand sides belong to $\mathbb{C}[[\zeta]]$ even if $\|\boldsymbol{\omega}\| = 0$, by the same argument as in the proof of Lemma 3.2, and in fact to $\mathbb{C}\{\zeta\}$, by induction on $r(\boldsymbol{\omega})$.

Since $\|\boldsymbol{\omega}\|$ always lies in $\mathbb{Z}$, we can apply Lemma 8.3: we get $\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}, \widehat{\mathcal{V}}_a^{\boldsymbol{\omega}} \in \widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ for all $\boldsymbol{\omega} \neq \emptyset$ by induction on $r(\boldsymbol{\omega})$, hence our moulds take their values in $\widehat{\boldsymbol{R}}_{\mathbb{Z}}$.

We see that the singularities of $\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}$ and $\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}$ are all simple singularities, because of the following addendum to Lemma 8.3: with the hypotheses and notations of that lemma, if moreover $\widehat{\varphi} \in \widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$, then $\widehat{b} * \widehat{\varphi}$ vanishes at the origin and only has simple singularities with vanishing residuum (this follows from the first formula in (8.14)), hence $\widehat{s}(\widehat{b} * \widehat{\varphi}) \in \widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$. □

Notice that, by iterating (9.6), one gets

$$\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}} = (-1)^r \frac{1}{\zeta - \hat{\omega}_1}\bigg(\hat{a}_{\omega_1} * \bigg(\frac{1}{\zeta - \hat{\omega}_2}\bigg(\hat{a}_{\omega_2} * \bigg(\cdots\bigg(\frac{1}{\zeta - \hat{\omega}_r}\hat{a}_{\omega_r}\bigg)\cdots\bigg)\bigg)\bigg)\bigg) \tag{9.7}$$

$$\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}} = \frac{1}{\zeta - \check{\omega}_r}\bigg(\hat{a}_{\omega_r} * \bigg(\frac{1}{\zeta - \check{\omega}_{r-1}}\bigg(\hat{a}_{\omega_{r-1}} * \bigg(\cdots\bigg(\frac{1}{\zeta - \check{\omega}_1}\hat{a}_{\omega_1}\bigg)\cdots\bigg)\bigg)\bigg)\bigg) \tag{9.8}$$

with the notation of (8.7). These are iterated integrals; for instance, the second formula can be written

$$\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}(\zeta) = \frac{1}{\zeta - \check{\omega}_r} \int\!\!\cdots\!\!\int_{0<\zeta_1<\cdots<\zeta_{r-1}<\zeta} \hat{a}_{\omega_r}(\zeta - \zeta_{r-1}) \cdot \\ \cdot \frac{\hat{a}_{\omega_{r-1}}(\zeta_{r-1} - \zeta_{r-2})}{\zeta_{r-1} - \check{\omega}_{r-1}} \cdots \frac{\hat{a}_{\omega_2}(\zeta_2 - \zeta_1)}{\zeta_2 - \check{\omega}_2} \frac{\hat{a}_{\omega_1}(\zeta_1)}{\zeta_1 - \check{\omega}_1}\, \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_{r-1} \tag{9.9}$$

and its analytic continuation along any parametrised path γ which starts from 0 and then avoids $\mathbb{Z}$ is given by the same integral, but taken over all $(r-1)$-tuples $(\zeta_1, \ldots, \zeta_{r-1}) = \big(\gamma(t_1), \ldots, \gamma(t_{r-1})\big)$ with $t_1 < \cdots < t_{r-1}$.

9.6. We are now ready to study the alien derivatives of the resurgent functions $\widetilde{\mathcal{V}}_a^{\omega_1,\ldots,\omega_r}$ and $\widetilde{\mathcal{V}}_a^{\omega_1,\ldots,\omega_r}$ (in the formal model as well as in the convolutive model, the difference is immaterial here).

Proposition 9.2. *Let $\Omega \subset \mathbb{Z}$ and $a = (\hat{a}_\eta)_{\eta \in \Omega}$ as in Lemma 9.1. For each $m \in \mathbb{Z}^*$, denote by the same symbol Δ_m the alien derivation of index m on $\boldsymbol{A} = \widetilde{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$ and the mould derivation it induces on $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ by (4.4). Then there exists a scalar-valued alternal mould $V_a^\bullet(m) \in \mathcal{M}^\bullet(\Omega, \mathbb{C})$ such that*

$$\Delta_m \widetilde{\mathcal{V}}_a^\bullet = \widetilde{\mathcal{V}}_a^\bullet \times V_a^\bullet(m), \quad \Delta_m \widetilde{\mathcal{V}}_a^\bullet = -V_a^\bullet(m) \times \widetilde{\mathcal{V}}_a^\bullet. \tag{9.10}$$

Moreover, if $\boldsymbol{\omega} \in \Omega^\bullet$ is non-empty,

$$\|\boldsymbol{\omega}\| \neq m \implies V_a^{\boldsymbol{\omega}}(m) = 0. \tag{9.11}$$

Proof. Since $\widetilde{\mathcal{V}}_a^\bullet$ and $S\widetilde{\mathcal{V}}_a^\bullet = \widetilde{\mathcal{V}}\!\!\!/_a^\bullet$ are mutually inverse, we get

$$\Delta_m \widetilde{\mathcal{V}}_a^\bullet = \widetilde{\mathcal{V}}_a^\bullet \times \widetilde{V}_a^\bullet(m), \quad \Delta_m \widetilde{\mathcal{V}}\!\!\!/_a^\bullet = \widetilde{V}\!\!\!/_a^\bullet(m) \times \widetilde{\mathcal{V}}\!\!\!/_a^\bullet$$

by *defining* the moulds $\widetilde{V}_a^\bullet(m)$ and $\widetilde{V}\!\!\!/_a^\bullet(m)$ as

$$\widetilde{V}_a^\bullet(m) = \widetilde{\mathcal{V}}\!\!\!/_a^\bullet \times \Delta_m \widetilde{\mathcal{V}}_a^\bullet, \quad \widetilde{V}\!\!\!/_a^\bullet(m) = \Delta_m \widetilde{\mathcal{V}}\!\!\!/_a^\bullet \times \widetilde{\mathcal{V}}_a^\bullet,$$

but a priori all these moulds take their values in $\boldsymbol{A}$. The operators Δ_m and S clearly commute, thus $\widetilde{V}\!\!\!/_a^\bullet(m) = S\widetilde{V}_a^\bullet(m)$. Proposition 5.4 shows that $\widetilde{V}_a^\bullet(m)$ and $\widetilde{V}\!\!\!/_a^\bullet(m)$ are alternal; Proposition 5.2 then shows that they are opposite of one another: $\widetilde{V}\!\!\!/_a^\bullet(m) = -\widetilde{V}_a^\bullet(m)$.

It only remains to be checked that $\widetilde{V}_a^\bullet(m)$ is scalar-valued and satisfies (9.11). This will follow from the equation

$$(\partial + \nabla - m)\widetilde{V}_a^\bullet(m) = 0, \tag{9.12}$$

where ∂ denotes the differential $\frac{\mathrm{d}}{\mathrm{d}z}$ as well as the mould derivation it induces by (4.4) and ∇ is the mould derivation (4.3).

Here is the proof of (9.12): $\widetilde{\mathcal{V}}_a^\bullet$ is defined on non-empty words by the first equation in (9.4), which can be written

$$(\partial + \nabla)\widetilde{\mathcal{V}}_a^\bullet = \widetilde{J}_a^\bullet \times \widetilde{\mathcal{V}}_a^\bullet, \tag{9.13}$$

with $\widetilde{J}_a^\bullet \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ defined exactly as in (4.6). Let us apply the derivation Δ_m to both sides of equation (9.13), using $\Delta_m(\partial + \nabla) = (\partial + \nabla - m)\Delta_m$ (consequence of (9.3) and of $[\nabla, \Delta_m] = 0$) and $\Delta_m \widetilde{J}_a^\bullet = 0$ (consequence of the vanishing of Δ_m on entire functions):

$$(\partial + \nabla - m)\Delta_m \widetilde{\mathcal{V}}_a^\bullet = \widetilde{J}_a^\bullet \times \Delta_m \widetilde{\mathcal{V}}_a^\bullet.$$

Writing $\Delta_m \widetilde{\mathcal{V}}_a^\bullet$ as $\widetilde{\mathcal{V}}_a^\bullet \times \widetilde{V}_a^\bullet(m)$ and using the fact that $\partial + \nabla$ is a derivation, we get

$$\big((\partial + \nabla)\widetilde{\mathcal{V}}_a^\bullet\big) \times \widetilde{V}_a^\bullet(m) + \widetilde{\mathcal{V}}_a^\bullet \times (\partial + \nabla - m)\widetilde{V}_a^\bullet(m) = \widetilde{J}_a^\bullet \times \widetilde{\mathcal{V}}_a^\bullet \times \widetilde{V}_a^\bullet(m),$$

whence $\widetilde{\mathcal{V}}_a^\bullet \times (\partial + \nabla - m)\widetilde{V}_a^\bullet(m) = 0$ by a further use of (9.13). Since $\widetilde{\mathcal{V}}_a^\emptyset = 1$ and $\mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ is an integral domain, this yields (9.12).

We conclude the proof by interpreting this relation in the convolutive model: we already knew that $\widetilde{V}_a^\emptyset(m) = 0$; now, for any non-empty $\boldsymbol{\omega}$, we have

$$\mathcal{B}\widetilde{V}_a^{\boldsymbol{\omega}}(m) = V_a^{\boldsymbol{\omega}}(m)\delta + \widehat{V}_a^{\boldsymbol{\omega}}(m)(\zeta)$$

with $V_a^{\boldsymbol{\omega}}(m) \in \mathbb{C}$ and $\widehat{V}_a^{\boldsymbol{\omega}}(m) \in \widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$ satisfying

$$(\|\boldsymbol{\omega}\| - m)V_a^{\boldsymbol{\omega}}(m) = 0, \quad (-\zeta + \|\boldsymbol{\omega}\| - m)\widehat{V}_a^{\boldsymbol{\omega}}(m) = 0,$$

whence $V_a^{\boldsymbol{\omega}}(m) = 0$ for $\|\boldsymbol{\omega}\| \neq m$ and $\widehat{V}_a^{\boldsymbol{\omega}}(m) = 0$ for all $\boldsymbol{\omega}$ (since both $\mathbb{C}$ and $\widehat{H}\big(\mathcal{R}(\mathbb{Z})\big) \subset \mathbb{C}\{\zeta\}$ are integral domains). $\square$

9.7. Formulas (9.10), when evaluated in the convolutive model on $\boldsymbol{\omega} \in \Omega^\bullet$, read

$$\Delta_m \widehat{\mathcal{V}}_a^{\emptyset} = \Delta_m \widehat{\mathcal{V}}_a^{\emptyset} = 0$$

for $r(\boldsymbol{\omega}) = 0$, which is obvious since $\widehat{\mathcal{V}}_a^{\emptyset} = \widehat{\mathcal{V}}_a^{\emptyset} = \delta$. For $r(\boldsymbol{\omega}) = 1$, we get

$$\Delta_m \widehat{\mathcal{V}}_a^{\omega_1} = -\Delta_m \widehat{\mathcal{V}}_a^{\omega_1} = V_a^{\omega_1}(m)\,\delta$$

and the explicit value of the coefficient is

$$\omega_1 = m \implies V_a^{\omega_1}(m) = -2\pi \mathrm{i}\, \hat{a}_m(m), \quad \omega_1 \neq m \implies V_a^{\omega_1}(m) = 0. \tag{9.14}$$

This is a simple residuum computation for the meromorphic functions $\widehat{\mathcal{V}}_a^{\omega_1}(\zeta) = -\widehat{\mathcal{V}}_a^{\omega_1}(\zeta) = -\frac{\hat{a}_{\omega_1}(\zeta)}{\zeta - \omega_1}$ (observe that the value of $\hat{a}_m$ at m and thus of $V_a^m(m)$ depend transcendentally on the Taylor coefficients of $\hat{a}_m$ at the origin).

For $r = r(\boldsymbol{\omega}) \geq 2$, we get

$$\begin{aligned} \Delta_m \widehat{\mathcal{V}}_a^{\boldsymbol{\omega}} &= V_a^{\boldsymbol{\omega}}(m)\,\delta + \sum_{i=1}^{r-1} V_a^{\omega_{i+1},\dots,\omega_r}(m)\,\widehat{\mathcal{V}}_a^{\omega_1,\dots,\omega_i}, \\ \Delta_m \widehat{\mathcal{V}}_a^{\boldsymbol{\omega}} &= -V_a^{\boldsymbol{\omega}}(m)\,\delta - \sum_{i=1}^{r-1} V_a^{\omega_1,\dots,\omega_i}(m)\,\widehat{\mathcal{V}}_a^{\omega_{i+1},\dots,\omega_r}. \end{aligned} \tag{9.15}$$

The number $V_a^{\boldsymbol{\omega}}(m)$ thus appears as the *residuum* of a certain simple singularity, which is a combination of the singularities at m of certain branches of $\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}$ or $\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}$; on the other hand, the fact that the *variation* of this singularity can be expressed as a linear combination of the functions $\widehat{\mathcal{V}}_a^{\omega_1,\dots,\omega_i}$ or $\widehat{\mathcal{V}}_a^{\omega_{i+1},\dots,\omega_r}$ is related to the very origin of the name "resurgent functions": the functions $\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}(\zeta)$ or $\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}(\zeta)$, which were initially defined for ζ close to the origin by (9.7)–(9.8), "resurrect" in the variation of the singularities of their analytic continuations.

An even more striking instance of this "resurgence phenomenon" is the Bridge Equation, to be discussed in the case of the saddle-node problem in Section 10 below.

9.8. The computation of the number $V_a^{\boldsymbol{\omega}}(m)$ is not as easy when $r(\boldsymbol{\omega}) \geq 2$ as in the case $r = 1$.

First observe that the vanishing of $V_a^{\boldsymbol{\omega}}(m)$ when $\|\boldsymbol{\omega}\| = \hat{\omega}_1 = \check{\omega}_r \neq m$ could be obtained as a consequence of the analytic continuation of formulas (9.7)–(9.8) (for instance, the singularities of the analytic continuation of $\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}$ can only be located at $\check{\omega}_1, \dots, \check{\omega}_r$ and, among them, only the one at $\check{\omega}_r$ can have a non-zero residuum – cf. the argument at the end of the proof of Lemma 9.1).

For $\|\boldsymbol{\omega}\| = m$, using the notations of Definition 9.2, one can write $V_a^{\boldsymbol{\omega}}(m)$ as a combination of iterated integrals: (9.9) and (9.15) yield

$$
\begin{aligned}
V_a^{\boldsymbol{\omega}}(m) = -2\pi\mathrm{i} \sum_{\varepsilon\in\{+,-\}^{|m|-1}} \frac{p_\varepsilon! q_\varepsilon!}{|m|!} \int_{\Gamma_\varepsilon} \hat{a}_{\omega_r}(m-\zeta_{r-1})\cdot \\
\cdot \frac{\hat{a}_{\omega_{r-1}}(\zeta_{r-1}-\zeta_{r-2})}{\zeta_{r-1}-\check{\omega}_{r-1}} \cdots \frac{\hat{a}_{\omega_2}(\zeta_2-\zeta_1)}{\zeta_2-\check{\omega}_2} \frac{\hat{a}_{\omega_1}(\zeta_1)}{\zeta_1-\check{\omega}_1}\, \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_{r-1},
\end{aligned}
\tag{9.16}
$$

where Γ_ε consists of all $(r-1)$-tuples $(\zeta_1,\dots,\zeta_{r-1}) = \big(\gamma_\varepsilon(t_1),\dots,\gamma_\varepsilon(t_{r-1})\big)$ with $t_1 < \cdots < t_{r-1}$, for any parametrisation of the oriented path γ_ε (which connects 0 and $m = \check{\omega}_r$). In fact, one can restrict oneself to the paths which follow the segment $]0,m[$ circumventing the points of $\{\check{\omega}_1,\dots,\check{\omega}_{r-1}\} \cap]0,m[= \{k_1,\dots,k_s\}$ to the right or to the left, labelled by sequences $\varepsilon \in \{+,-\}^s$, with weights $p_\varepsilon! q_\varepsilon!/(s+1)!$.

The formula gets simpler when $\Omega \subset \mathbb{Z}^*$ and $\tilde{a}_\eta \equiv z^{-1}$ for each $\eta \in \Omega$, since each $\hat{a}_\eta$ is then the constant function with value 1:

$$
\|\boldsymbol{\omega}\| = m \implies V_a^{\boldsymbol{\omega}}(m) = -2\pi\mathrm{i} \sum_{\varepsilon\in\{+,-\}^{|m|-1}} \frac{p_\varepsilon! q_\varepsilon!}{|m|!} \int_{\Gamma_\varepsilon} \frac{\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{r-1}}{(\zeta_1-\check{\omega}_1)\cdots(\zeta_{r-1}-\check{\omega}_{r-1})}.
$$

In this last case, the numbers $V_a^{\boldsymbol{\omega}}(m)$ are connected with multiple logarithms. They are studied under the name "canonical hyperlogarithmic mould" in [3], chap. 7, without the restriction $\Omega \subset \mathbb{Z}$ (which we imposed here only to avoid having to define the larger algebra $\widehat{\mathrm{RES}}^{\mathrm{simp}}$; also the condition $0 \notin \Omega$ was imposed here only to simplify the discussion).

Observe that $V_a^\bullet(m)$ is always a primitive element of the graded cocommutative Hopf algebra $\mathcal{H}^\bullet(\Omega,\mathbb{C})$ defined in Section 5 (this is just a rephrasing of the shuffle relations encoded by the alternality of this scalar mould).

9.9. Formulas (9.10) can be iterated so as to express all the successive alien derivatives of our resurgent functions $\tilde{\mathcal{V}}_a^{\boldsymbol{\omega}}$ or $\tilde{\mathcal{V}}_a^{\boldsymbol{\omega}}$:

$$
\begin{aligned}
\Delta_{m_s}\cdots\Delta_{m_1}\tilde{\mathcal{V}}_a^\bullet &= \tilde{\mathcal{V}}_a^\bullet \times V_a^\bullet(m_s)\times\cdots\times V_a^\bullet(m_1),\\
\Delta_{m_s}\cdots\Delta_{m_1}\tilde{\mathcal{V}}_a^\bullet &= (-1)^s V_a^\bullet(m_1)\times\cdots\times V_a^\bullet(m_s)\times\tilde{\mathcal{V}}_a^\bullet,
\end{aligned}
\tag{9.17}
$$

for $s \ge 1$ and $m_1,\dots,m_s \in \mathbb{Z}^*$.

We can consider the collection of resurgent functions $(\tilde{\mathcal{V}}_a^{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet}$ (or $(\tilde{\mathcal{V}}_a^{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet}$) as closed under alien derivation (i.e. all their alien derivatives can be expressed through relations involving themselves and scalars); it was already closed under multiplication (by symmetrality), and even under ordinary differentiation, in view of (9.4), if we admit relations with coefficients in $\mathbb{C}\{z^{-1}\}$ (but, after all, convergent series can be considered as "resurgent constants": all alien derivations act trivially on them).

This is why the $\tilde{\mathcal{V}}_a^{\boldsymbol{\omega}}$'s are called "resurgent monomials": they behave nicely under elementary operations such as multiplication and alien derivations. In fact, in Section 12 below, we shall deduce from them another family of resurgence monomials which behave even better under the action of alien derivations (but the price to pay is that their ordinary derivatives are not as simple as (9.4)).

Notice that the operator $\Delta_{m_s}\cdots\Delta_{m_1}$ measures a combination of singularities located at $m_1+\cdots+m_s$. For instance, the fact that $V_a^\bullet(m_s)\times\cdots\times V_a^\bullet(m_1)$ vanishes on any word $\boldsymbol{\omega}$ such that $\|\boldsymbol{\omega}\|\neq m_1+\cdots+m_s$ (easy consequence of (9.11)) is consistent with the vanishing of the residuum at any point $\neq\hat{\omega}_1$ of any branch of $\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}$ (consequence of the analytic continuation of (9.7)).

9.10. Let $\Omega\subset\mathbb{Z}$ and $a=(\hat{a}_\eta)_{\eta\in\Omega}$ be a family of entire functions as in Lemma 9.1, thus with $\hat{a}_0(0)=0$ if $0\in\Omega$. We end this section by illustrating mould calculus to derive quadratic shuffle relations for the numbers

$$L_a^{\boldsymbol{\omega}}=2\pi\mathrm{i}\int_{\Gamma^+}\hat{a}_{\omega_r}(\|\boldsymbol{\omega}\|-\zeta_{r-1})\frac{\hat{a}_{\omega_{r-1}}(\zeta_{r-1}-\zeta_{r-2})}{\zeta_{r-1}-\check{\omega}_{r-1}}\cdots\frac{\hat{a}_{\omega_2}(\zeta_2-\zeta_1)}{\zeta_2-\check{\omega}_2}\frac{\hat{a}_{\omega_1}(\zeta_1)}{\zeta_1-\check{\omega}_1}\,\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{r-1}, \tag{9.18}$$

for $\boldsymbol{\omega}\in\Omega^\bullet$ non-empty, where $\Gamma^+=\Gamma_\varepsilon$ with $\varepsilon=(+,\dots,+)\in\{+,-\}^{|m|-1}$ for $m=\|\boldsymbol{\omega}\|$ (notation of (9.16); if $r=1$, then $L_a^{\omega_1}=2\pi\mathrm{i}\,\hat{a}_{\omega_1}(\omega_1)$). This includes the case of the multiple logarithms

$$L^{\boldsymbol{\omega}}=2\pi\mathrm{i}\int_{\Gamma^+}\frac{\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{r-1}}{(\zeta_1-\check{\omega}_1)\cdots(\zeta_{r-1}-\check{\omega}_{r-1})}, \tag{9.19}$$

with $\omega_1,\dots,\omega_r\in\Omega\subset\mathbb{Z}^*$ (obtained when $\hat{a}_\eta(\zeta)\equiv 1$).[12]

It is convenient to use here the auxiliary operators Δ_m^+ of $\widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$ defined by the formulas $\Delta_0^+=\mathrm{Id}$ and, for $m\in\mathbb{Z}^*$,

$$\Delta_m^+(c\,\delta+\widehat{\varphi})=\mathrm{sing}_m(\mathrm{cont}_{\gamma^+}\widehat{\varphi}), \tag{9.20}$$

where $\gamma^+=\gamma_\varepsilon$ with $\varepsilon=(+,\dots,+)\in\{+,-\}^{|m|-1}$. Thus

$$L_a^{\boldsymbol{\omega}}=\text{coefficient of }\delta\text{ in }\Delta_{\|\boldsymbol{\omega}\|}^+\widehat{\mathcal{V}}_a^{\boldsymbol{\omega}}. \tag{9.21}$$

We shall consider $L_a^{\boldsymbol{\omega}}$ as the value at $\boldsymbol{\omega}$ of a scalar mould $L_a^\bullet$; we set $L_a^{\emptyset}=1$, so that (9.21) still holds when $\boldsymbol{\omega}=\emptyset$.

Proposition 9.3. *The numbers $L_a^{\boldsymbol{\omega}}$ satisfy the shuffle relations*

$$\sum_{\boldsymbol{\omega}\in\Omega^\bullet}\mathrm{sh}\binom{\boldsymbol{\omega}^1,\,\boldsymbol{\omega}^2}{\boldsymbol{\omega}}L_a^{\boldsymbol{\omega}}=\begin{cases}L_a^{\boldsymbol{\omega}^1}L_a^{\boldsymbol{\omega}^2} & \textit{if } \|\boldsymbol{\omega}^1\|\cdot\|\boldsymbol{\omega}^2\|\geq 0,\\ 0 & \textit{if not,}\end{cases}$$

for any non-empty $\boldsymbol{\omega}^1,\boldsymbol{\omega}^2\in\Omega^\bullet$. Equivalently, the scalar moulds $L_{a,\pm}^\bullet$ defined by

$$L_{a,+}^{\boldsymbol{\omega}}=1_{\{\|\boldsymbol{\omega}\|\geq 0\}}L_a^{\boldsymbol{\omega}},\quad L_{a,-}^{\boldsymbol{\omega}}=1_{\{\|\boldsymbol{\omega}\|\leq 0\}}L_a^{\boldsymbol{\omega}} \tag{9.22}$$

(for any $\boldsymbol{\omega}\in\Omega^\bullet$, with the convention $\|\emptyset\|=0$) are symmetral.

[12] We recall that $\check{\omega}_1=\omega_1,\check{\omega}_2=\omega_1+\omega_2,\dots,\check{\omega}_{r-1}=\omega_1+\cdots+\omega_{r-1}$ (thus $L^{\boldsymbol{\omega}}$ depends on ω_r only through Γ^+ which connects the origin and $\check{\omega}_r$).

This can be rephrased by saying that $L^\bullet_{a,+}$ and $L^\bullet_{a,-}$ are group-like elements of the graded cocommutative Hopf algebra $\mathcal{H}^\bullet(\Omega,\mathbb{C})$ defined in Section 5.

The rest of this section is devoted to the proof of Proposition 9.3. We begin by a few facts about the operators Δ^+_m; these are not derivations, as the alien derivations Δ_m, but they are related to them and satisfy modified Leibniz rules analogous to (7.4):

Lemma 9.2. *The operators Δ^+_m defined in (9.20) are related to the alien derivations (9.2) by the following relations: for any $m \in \mathbb{Z}^*$,*

$$\Delta^+_m = \sum \tfrac{1}{s!}\Delta_{m_s}\cdots\Delta_{m_1}, \qquad \Delta_m = \sum \tfrac{(-1)^{s-1}}{s}\Delta^+_{m_s}\cdots\Delta^+_{m_1}, \tag{9.23}$$

with both sums taken over all $s \geq 1$ and $m_1,\dots,m_s \in \mathbb{Z}^*$ of the same sign as m such that $m_1+\cdots+m_s = m$ (these are thus finite sums). Moreover, for any $\widehat{\chi}_1,\widehat{\chi}_2 \in \widehat{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ and $m \in \mathbb{Z}$,

$$\Delta^+_m(\widehat{\chi}_1 * \widehat{\chi}_2) = \sum \Delta^+_{m_1}\widehat{\chi}_1 * \Delta^+_{m_2}\widehat{\chi}_2 \tag{9.24}$$

with summation over all $m_1,m_2 \in \mathbb{Z}$ of the same sign as m (but possibly vanishing) such that $m_1+m_2 = m$.

Let us denote by the same symbols the operators of $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ obtained from the Δ^+_m's by conjugacy by the formal Borel transform $\mathcal{B}$, as we did for the Δ_m's. If we consider the algebras $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}[[\mathrm{e}^{-z}]]$ and $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}[[\mathrm{e}^{z}]]$, formula (9.23) can be written

$$\sum_{m\geq 0}\mathrm{e}^{-mz}\Delta^+_m = \exp\Big(\sum_{m>0}\mathrm{e}^{-mz}\Delta_m\Big), \qquad \sum_{m\leq 0}\mathrm{e}^{-mz}\Delta^+_m = \exp\Big(\sum_{m<0}\mathrm{e}^{-mz}\Delta_m\Big). \tag{9.25}$$

We do not give the proof of this lemma here; see e.g. [14], Lemmas 4 and 5 (the coefficients $p_\varepsilon! q_\varepsilon!/|m|!$ in Definition 9.2 were chosen exactly so that (9.23) hold; the standard properties of the logarithm and exponential series then show that (9.24) and Proposition 9.1 are equivalent; it is in fact easy to check first (9.24) by deforming the contour of integration in the integral giving $\widehat{\chi}_1 * \widehat{\chi}_2$, and then to deduce Proposition 9.1).

Lemma 9.3. *For any $m \in \mathbb{Z}^*$, define a scalar mould $L^\bullet_a(m)$ by the formula*

$$L^\bullet_a(m) = \sum \tfrac{(-1)^s}{s!} V^\bullet_a(m_1)\times\cdots\times V^\bullet_a(m_s),$$

with summation over all $s \geq 1$ and $m_1,\dots,m_s \in \mathbb{Z}^*$ of the same sign as m such that $m_1+\cdots+m_s = m$. Define also $L^\bullet_a(0) = 1^\bullet$. Then, for every $m \in \mathbb{Z}$,

(i) $\Delta^+_m \widehat{\mathcal{V}}^\bullet_a = L^\bullet_a(m)\times\widehat{\mathcal{V}}^\bullet_a$,

(ii) $\tau\big(L^\bullet_a(m)\big) = \sum L^\bullet_a(m_1)\otimes L^\bullet_a(m_2)$, *with summation over all $m_1,m_2 \in \mathbb{Z}$ of the same sign as m such that $m_1+m_2 = m$,*

(iii) $m = \|\boldsymbol{\omega}\| \implies L^{\boldsymbol{\omega}}_a(m) = L^{\boldsymbol{\omega}}_a$, $m \neq \|\boldsymbol{\omega}\| \implies L^{\boldsymbol{\omega}}_a(m) = 0$ *(for any $\boldsymbol{\omega} \in \Omega^\bullet$, with the convention $\|\emptyset\| = 0$).*

Proof. The first property follows from (9.17) and (9.23). For the second, we write the symmetrality of $\widehat{\mathcal{V}}^\bullet_a$ and $\widehat{\mathcal{V}}^\bullet_a$ as identities in $\mathcal{M}^{\bullet\bullet}(\Omega, \widehat{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}})$:

$$\tau(\widehat{\mathcal{V}}^\bullet_a) = \widehat{\mathcal{V}}^\bullet_a \otimes \widehat{\mathcal{V}}^\bullet_a, \quad \tau(\widehat{\mathcal{V}}^\bullet_a) = \widehat{\mathcal{V}}^\bullet_a \otimes \widehat{\mathcal{V}}^\bullet_a,$$

the operator Δ^+_m induces operators acting on moulds and dimoulds which clearly satisfy $\Delta^+_m \circ \tau(\widehat{\mathcal{V}}^\bullet_a) = \tau(\Delta^+_m \widehat{\mathcal{V}}^\bullet_a)$ and relation (9.24) implies

$$\tau(\Delta^+_m \widehat{\mathcal{V}}^\bullet_a) = \sum_{\substack{m=m_1+m_2\\ m_i m \geq 0}} \Delta^+_{m_1} \widehat{\mathcal{V}}^\bullet_a \otimes \Delta^+_{m_2} \widehat{\mathcal{V}}^\bullet_a,$$

whence the result follows since $\tau\big(L^\bullet_a(m)\big) = \tau(\Delta^+_m \widehat{\mathcal{V}}^\bullet_a) \times \tau(\widehat{\mathcal{V}}^\bullet_a)$ by the homomorphism property of τ applied to (i).

The second part of the third property is obvious when $m = 0$ and follows from (9.11) when $m \neq 0$, because $\|\boldsymbol{\omega}\| \neq m_1 + \cdots + m_s$ implies that $V^\bullet_a(m_1) \times \cdots \times V^\bullet_a(m_s)$ vanishes on $\boldsymbol{\omega}$ (even if $\boldsymbol{\omega} = \emptyset$). The first part of the third property follows from (9.21), since property (i) yields

$$\Delta^+_m \widehat{\mathcal{V}}^{\boldsymbol{\omega}}_a = L^{\boldsymbol{\omega}}_a(m)\,\delta + \sum_{i=1}^{r-1} L^{\omega_1,\dots,\omega_i}_a(m) \widehat{\mathcal{V}}^{\omega_{i+1},\dots,\omega_r}_a$$

if $r = r(\boldsymbol{\omega}) \geq 1$ and $\Delta^+_m \widehat{\mathcal{V}}^{\emptyset}_a = L^{\emptyset}_a(m)\delta$ if $\boldsymbol{\omega} = \emptyset$. □

Proof of Proposition 9.3. We have $L^{\emptyset}_a = 1$. Let $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \in \Omega^\bullet$ be non-empty. Property (ii) with $m = \|\boldsymbol{\omega}^1\| + \|\boldsymbol{\omega}^2\|$ yields

$$\sum_{\boldsymbol{\omega} \in \Omega^\bullet} \mathrm{sh}\binom{\boldsymbol{\omega}^1,\ \boldsymbol{\omega}^2}{\boldsymbol{\omega}} L^{\boldsymbol{\omega}}_a(m) = \sum_{\substack{m=m_1+m_2\\ m_i m \geq 0}} L^{\boldsymbol{\omega}^1}_a(m_1) L^{\boldsymbol{\omega}^2}_a(m_2).$$

According to Property (iii), the left-hand side is $\tau(L^\bullet_a)^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2}$ (because any nonzero term in it has $\|\boldsymbol{\omega}\| = m$). Among the $|m| + 1$ terms of the right-hand side, at most one may be nonzero: if $\|\boldsymbol{\omega}^1\|$ and $\|\boldsymbol{\omega}^2\|$ have the same sign, then the term corresponding to $m_1 = \|\boldsymbol{\omega}^1\|$ is $L^{\boldsymbol{\omega}^1}_a L^{\boldsymbol{\omega}^2}_a$ while all the others vanish; but in the opposite case, this term does not belong to the summation and one gets 0 as right-hand side. This is the desired shuffle relation; we leave it to the reader to interpret it in terms of symmetrality for the moulds $L^\bullet_{a,\pm}$ by distinguishing the four possible cases: $\|\boldsymbol{\omega}^1\| \cdot \|\boldsymbol{\omega}^2\| \geq 0$ or < 0, and $\|\boldsymbol{\omega}^1\| + \|\boldsymbol{\omega}^2\| \geq 0$ or < 0. □

In fact, we can write

$$L^\bullet_{a,+} = \sum_{m \geq 0} L^\bullet_a(m) = \exp\big(-V^\bullet_{a,+}\big), \quad V^\bullet_{a,+} = \sum_{m>0} V^\bullet_a(m) \tag{9.26}$$

$$L^\bullet_{a,-} = \sum_{m \leq 0} L^\bullet_a(m) = \exp\big(-V^\bullet_{a,-}\big), \quad V^\bullet_{a,-} = \sum_{m<0} V^\bullet_a(m), \tag{9.27}$$

with well-defined alternal moulds $V^\bullet_{a,\pm}$ (and using exp as a short-hand for E_1 – see (4.9)), since Lemma 9.3 (iii) and property (9.11) imply that, when evaluated on a given word $\boldsymbol{\omega}$, these formulas involve only finitely many terms; one could thus have invoked Proposition 5.1 to deduce the symmetrality of $L^\bullet_{a,\pm}$.

10 The Bridge Equation for the saddle-node

In this section, returning to the saddle-node problem, we shall explain why the formal series $\widetilde{\varphi}_n(z) = \varphi_n(-1/z)$ and $\widetilde{\psi}_n(z) = \psi_n(-1/z)$ of Theorem 2, which were proved to belong to $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$, are in fact simple resurgent functions. Moreover, we shall express their alien derivatives in terms of themselves and of the numbers $V^\bullet_a(m)$ of Proposition 9.2.

10.1. We recall the hypotheses and the notations for the saddle-node:

$$X = x^2\frac{\partial}{\partial x} + A(x,y)\frac{\partial}{\partial y}$$

with $A(x,y) = y + \sum_{\eta\in\Omega} a_\eta(x) y^{\eta+1} \in \mathbb{C}\{x,y\}$, where $\Omega = \{\,\eta \in \mathbb{Z} \mid \eta \geq -1\,\}$, $\widetilde{a}_\eta(z) = a_\eta(-1/z) \in z^{-1}\mathbb{C}\{z^{-1}\}$ and $\widetilde{a}_0(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$.

We also recall that $\eta \in \Omega \mapsto B_\eta = y^{\eta+1}\frac{\partial}{\partial y}$ gives rise to a comould $\boldsymbol{B}_\bullet$ such that $\boldsymbol{B}_{\boldsymbol{\omega}} y = \beta_{\boldsymbol{\omega}} y^{\|\boldsymbol{\omega}\|+1}$, where the numbers $\beta_{\boldsymbol{\omega}}$, $\boldsymbol{\omega} \in \Omega^\bullet$, satisfy Lemma 8.1 (we define $\beta_\emptyset = 1$ and $\|\emptyset\| = 0$). We set $a = (\mathcal{B}\widetilde{a}_\eta)_{\eta\in\Omega}$, so as to be able to make use of the constants $V^{\boldsymbol{\omega}}_a(m)$, $(m,\boldsymbol{\omega}) \in \mathbb{Z}^* \times \Omega^\bullet$ defined in Proposition 9.2 and more explicitly by formulas (9.14) and (9.16). Later in this section we shall prove

Proposition 10.1. *The family of complex numbers $\big(\beta_{\boldsymbol{\omega}} V^{\boldsymbol{\omega}}_a(m)\big)_{\boldsymbol{\omega}\in\Omega^\bullet,\,\|\boldsymbol{\omega}\|=m}$ is summable for each $m \in \mathbb{Z}^*$. Let*

$$C_m = \sum_{\boldsymbol{\omega}\in\Omega^\bullet,\,\|\boldsymbol{\omega}\|=m} \beta_{\boldsymbol{\omega}} V^{\boldsymbol{\omega}}_a(m), \quad m \in \mathbb{Z}^*. \tag{10.1}$$

Then $C_m = 0$ for $m \leq -2$.

We call *Écalle's invariants* of X the complex numbers $C_{-1}, C_1, C_2, \dots, C_m, \dots$ because of their role in the Bridge Equation (Theorem 3 below) and in the classification problem (Theorem 5 and Section 11 below).

The formal transformations $\theta(x,y) = \big(x, \varphi(x,y)\big)$ and $\theta^{-1}(x,y) = \big(x, \psi(x,y)\big)$ which conjugate X to its normal form $X_0 = x^2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ were constructed in the first part of this article through mould-comould expansions for the corresponding substitution operators Θ and Θ^{-1}. Passing to the resurgence variable $z = -1/x$, we set

$$\widetilde{\varphi}(z,y) = \varphi(-1/z,y) = y + \sum_{n\geq 0} \widetilde{\varphi}_n(z) y^n,$$

$$\widetilde{\psi}(z,y) = \psi(-1/z,y) = y + \sum_{n\geq 0} \widetilde{\psi}_n(z) y^n,$$

where the coefficients $\widetilde{\varphi}_n(z)$ and $\widetilde{\psi}_n(z)$ are known to belong to the algebra $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$ of resurgent functions, by Theorem 2. We also introduce the substitution operator

$$\widetilde{\Theta}\colon\ \tilde{f}(z,y) \mapsto \tilde{f}\big(z,\widetilde{\varphi}(z,y)\big) \tag{10.2}$$

(a priori defined in $\mathbb{C}[[z^{-1},y]]$). Later in this section, we shall prove

Theorem 3. *The formal series $\widetilde{\varphi}_n(z)$ and $\widetilde{\psi}_n(z)$ are simple resurgent functions, thus $\widetilde{\varphi}(z,y)$ and $\widetilde{\psi}(z,y)$ belong in fact to $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}[[y]]$.*

Moreover, for any $m \in \mathbb{Z}^*$, the formal series of $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}[[y]]$

$$\Delta_m\widetilde{\varphi} := \sum_{n\geq 0} (\Delta_m\widetilde{\varphi}_n) y^n, \quad \Delta_m\widetilde{\psi} := \sum_{n\geq 0} (\Delta_m\widetilde{\psi}_n) y^n$$

are given by the formulas

$$\Delta_m\widetilde{\varphi} = C_m y^{m+1} \frac{\partial\widetilde{\varphi}}{\partial y}, \quad \Delta_m\widetilde{\psi} = -C_m \widetilde{\psi}^{m+1}, \quad m \in \mathbb{Z}^*. \tag{10.3}$$

10.2. The two equations in (10.3) are equivalent forms of the so-called *Bridge Equation*, here expressed in $\boldsymbol{A}[[y]]$ with $\boldsymbol{A} = \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$. On the one hand, the left-hand sides represent the action of the alien derivation Δ_m of $\boldsymbol{A}[[y]]$ (we denote by the same symbol the alien derivation Δ_m of $\boldsymbol{A}$ and the operator it induces in $\boldsymbol{A}[[y]]$ by acting separately on each coefficient). On the other hand, both right-hand sides can be expressed with the help of the ordinary differential operator

$$\mathcal{C}(m) = C_m y^{m+1} \frac{\partial}{\partial y},$$

yielding

$$\Delta_m\widetilde{\varphi} = \mathcal{C}(m)\widetilde{\varphi} = \mathcal{C}(m)\widetilde{\Theta} y, \tag{10.4}$$

$$\Delta_m\widetilde{\psi} = -\widetilde{\Theta}^{-1}\mathcal{C}(m) y. \tag{10.5}$$

See the end of this section for more symmetric formulations of the Bridge Equation, which involve only the operators $\widetilde{\Theta}$ or $\widetilde{\Theta}^{-1}$ and Δ_m for the left-hand sides, and $\mathcal{C}(m)$ for the right-hand sides.

The name "Bridge Equation" refers to the link thus established between alien and ordinary differential calculus when dealing with the solutions $\widetilde{\varphi}$ and $\widetilde{\psi}$ of our formal normalisation problem (or with the operator Θ solution of the conjugacy equation (3.1)).

This is a very general phenomenon, in which one sees the advantage of measuring the singularities in the Borel plane though *derivations*: we are dealing with the solutions of non-linear equations (e.g. $(\partial + y\frac{\partial}{\partial y})\widetilde{\varphi}(z,y) = A(-1/z,\widetilde{\varphi}(z,y))$ in $\mathbb{C}[[z^{-1},y]]$), and their alien derivatives must satisfy equations corresponding to the linearisation of these equations; its is thus natural that these alien derivatives can be expressed in terms of the ordinary derivatives of the solutions.

The above argument could be used to derive the form of equation (10.3)[13], however, in the proof below, we prefer to use the explicit mould representations involving $\widetilde{\mathcal{V}}^\bullet$ and $\widetilde{\mathcal{V}}^\bullet$ so as to obtain formulas (10.1) for the coefficients C_m.

10.3. Theorem 3 could also have been formulated in terms of the formal integral defined by (2.6): $\widetilde{Y}(z,u) = \widetilde{\varphi}(z,u\,\mathrm{e}^z) \in \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}[[u\,\mathrm{e}^z]]$ and

$$\dot{\Delta}_m \widetilde{Y} = C_m u^{m+1}\frac{\partial \widetilde{Y}}{\partial u}, \quad m \in \mathbb{Z}^*,$$

where $\dot{\Delta}_m = \mathrm{e}^{-mz}\Delta_m$ is the *dotted alien derivation of index m*, which already appeared in formula (9.25).

10.4. The Bridge Equations (10.3) are a compact writing of infinitely many "resurgence equations" for the series $\Delta_m\widetilde{\varphi}_n$ or $\Delta_m\widetilde{\psi}_n$, obtained by expanding them in powers of y.

For instance, setting

$$\widetilde{\Phi}_n = \begin{cases} 1+\widetilde{\varphi}_1 & \text{if } n=1, \\ \widetilde{\varphi}_n & \text{if } n \neq 1, \end{cases} \tag{10.6}$$

so that $\widetilde{\varphi}(z,y) = \sum_{n\ge 0}\widetilde{\Phi}_n(z)y^n$, we get

$$\Delta_m\widetilde{\Phi}_n = \begin{cases} (n-m)C_m\widetilde{\Phi}_{n-m} & \text{if } -1 \le m \le n-1, \\ 0 & \text{if } m \le -2 \text{ or } m \ge n. \end{cases}$$

Thus

- $\Delta_m\widetilde{\varphi}_0 = 0$ for $m \neq -1$, while $\Delta_{-1}\widetilde{\varphi}_0 = C_{-1}(1+\widetilde{\varphi}_1)$;
- $\Delta_m\widetilde{\varphi}_1 = 0$ for $m \neq -1$, while $\Delta_{-1}\widetilde{\varphi}_1 = 2C_{-1}\widetilde{\varphi}_2$;

[13] Compare the linear equations $L\partial_y\widetilde{\varphi} = 0$ and $(L-m-1)\Delta_m\widetilde{\varphi} = 0$ where

$$L = \widetilde{X}_0 + \tilde{\lambda}(z,y), \quad \tilde{\lambda}(z,y) = 1 - \partial_y A(-1/z,\widetilde{\varphi}(z,y)), \quad \widetilde{X}_0 = \partial + y\frac{\partial}{\partial y}.$$

The second equation follows from (9.3) for the computation of $\Delta_m(\partial + y\frac{\partial}{\partial y})\widetilde{\varphi}(z,y)$, and from the relation $\Delta_m A(-1/z,\widetilde{\varphi}(z,y)) = \big(\partial_y A(-1/z,\widetilde{\varphi}(z,y))\big)\Delta_m\widetilde{\varphi}(z,y)$ deduced from Proposition 10.2 below (indeed, $A(-1/z,y) \in \mathbb{C}\{z^{-1},y\} \subset \boldsymbol{A}\{y\}$). Since $\partial_y\widetilde{\varphi} = 1 + \mathcal{O}(z^{-1},y)$ is invertible, we can set $\widetilde{\chi} = (\partial_y\widetilde{\varphi})^{-1}\Delta_m\widetilde{\varphi}$; the above linear equations imply that $\widetilde{\chi}$ is annihilated by $\widetilde{X}_0 - (m+1)$, thus proportional to y^{m+1}: there exists $c_m \in \mathbb{C}$ such that $\Delta_m\widetilde{\varphi} = c_m y^{m+1}\partial_y\widetilde{\varphi}(z,y)$.

The relation $\Delta_m\widetilde{\psi} = -c_m\widetilde{\psi}^{m+1}$ follows by the alien chain rule: $y = \widetilde{\varphi}(z,\widetilde{\psi}(z,y)) = \widetilde{\Theta}^{-1}\widetilde{\varphi}$ implies $(\Delta_m\widetilde{\varphi})(z,\widetilde{\psi}) + \partial_y\widetilde{\varphi}(z,\widetilde{\psi})\Delta_m\widetilde{\psi} = 0$ by Proposition 10.2 below (using $\widetilde{\varphi} \in \boldsymbol{A}\{y\}$).

- $\Delta_m\widetilde{\varphi}_2 = 0$ for $m \notin \{-1,1\}$, while $\Delta_{-1}\widetilde{\varphi}_2 = 3C_{-1}\widetilde{\varphi}_3$ and $\Delta_1\widetilde{\varphi}_2 = C_1(1+\widetilde{\varphi}_1)$;
- $\Delta_m\widetilde{\varphi}_3 = 0$ for $m \notin \{-1,1,2\}$, while…

$\vdots$

Similarly, with

$$\widetilde{\Psi}_n = \begin{cases} 1+\widetilde{\psi}_1 & \text{if } n=1,\\ \widetilde{\psi}_n & \text{if } n\neq 1,\end{cases}$$

we have $\sum(\Delta_m\widetilde{\Psi}_n)y^n = -C_m\big(\sum\widetilde{\Psi}_n y^n\big)^{m+1}$, which means that $\Delta_m\widetilde{\Psi}_n = 0$ for all $n\in\mathbb{N}$ when $m\le -2$,

$$\Delta_{-1}\widetilde{\Psi}_n = \begin{cases} -C_{-1} & \text{if } n=0,\\ 0 & \text{if } n\neq 0\end{cases}$$

and $\Delta_m\widetilde{\Psi}_n = -C_m\sum_{n_1+\ldots+n_{m+1}=n}\widetilde{\Psi}_{n_1}\cdots\widetilde{\Psi}_{n_{m+1}}$ for any $n\in\mathbb{N}$ and $m\ge 1$.

In particular, C_m is the constant term in $\Delta_m\widetilde{\varphi}_{m+1}$ or in $-\Delta_m\widetilde{\psi}_{m+1}$.

10.5. *Proof of Proposition 10.1 and Theorem 3.* We have a Fréchet space structure on $\widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$, with seminorms $\|\,.\,\|_K$ indexed by the compact subsets of $\mathcal{R}(\mathbb{Z})$:

$$\|\widehat{\varphi}\|_K = \max_{\zeta\in K}|\widehat{\varphi}(\zeta)|\,,\quad \widehat{\varphi}\in\widehat{H}\big(\mathcal{R}(\mathbb{Z})\big),\ K\in\mathcal{K}.$$

We thus naturally get Fréchet space structures on $\widehat{\boldsymbol{R}}_{\mathbb{Z}} = \mathbb{C}\,\delta\oplus\widehat{H}\big(\mathcal{R}(\mathbb{Z})\big)$, by defining $\|c\,\delta+\widehat{\varphi}\|_K := \max\big(|c|,\|\widehat{\varphi}\|_K\big)$, and on $\widetilde{\boldsymbol{R}}_{\mathbb{Z}} = \mathcal{B}^{-1}\widehat{\boldsymbol{R}}_{\mathbb{Z}}$, with $\|\widetilde{\chi}\|_K := \|\mathcal{B}\widetilde{\chi}\|_K$ for $\widetilde{\chi} = c+\widetilde{\varphi}\in\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$.

The space $\boldsymbol{A} = \widetilde{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$ of simple resurgent functions is a closed subspace of $\widetilde{\boldsymbol{R}}_{\mathbb{Z}}$ and the Δ_m are continuous operators. Indeed, the map $\widehat{\varphi}\mapsto \mathrm{sing}_m(\mathrm{cont}_\gamma\,\widehat{\varphi})$ is continuous on $\hat{\boldsymbol{A}} = \widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$ because the variation can be expressed as a difference of branches and the residuum as a Cauchy integral.

Consider now the formal series $\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}(z) = \widetilde{\mathcal{V}}_a^{\boldsymbol{\omega}}(z), \widetilde{\mathcal{V}}^{\boldsymbol{\omega}}(z) = \widetilde{\mathcal{V}}_a^{\boldsymbol{\omega}}(z)\in\boldsymbol{A}$, and their formal Borel transforms, which belong to $\hat{\boldsymbol{A}}$. The end of the proof of Theorem 2 shows that $(\beta_{\boldsymbol{\omega}}\widehat{\mathcal{V}}^{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet,\,\|\boldsymbol{\omega}\|=n-1}$ and $(\beta_{\boldsymbol{\omega}}\widehat{\mathcal{V}}^{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet,\,\|\boldsymbol{\omega}\|=n-1}$ are summable families of $\hat{\boldsymbol{A}}$ for each $n\in\mathbb{N}$; indeed, for any compact subset K of $\mathcal{R}(\mathbb{Z})$, there exist ρ, N and L such that any point of K is the endpoint of a $(\rho,N,n-\mathbb{N}^*)$-adapted path of length $\le L$ and also the endpoint of a $(\rho,N,\mathbb{N})$-adapted path of length $\le L$, and one can use (8.24), (8.27) and (8.28). Hence the sums $\widehat{\varphi}_n$ and $\widehat{\psi}_n$ of these families belong to $\hat{\boldsymbol{A}}$. Equivalently, the formal series $\widetilde{\varphi}_n$ and $\widetilde{\psi}_n$ appear as sums of summable families of $\boldsymbol{A}$:

$$\widetilde{\varphi}_n = \sum_{\boldsymbol{\omega}\in\Omega^\bullet,\,\|\boldsymbol{\omega}\|=n-1}\beta_{\boldsymbol{\omega}}\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}\quad\text{and}\quad \widetilde{\psi}_n = \sum_{\boldsymbol{\omega}\in\Omega^\bullet,\,\|\boldsymbol{\omega}\|=n-1}\beta_{\boldsymbol{\omega}}\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}\quad\text{in } \boldsymbol{A},$$

they are thus simple resurgent functions themselves. To end the proof of Theorem 3, we thus only have to study the alien derivatives $\Delta_m\widetilde{\varphi}_n$ and $\Delta_m\widetilde{\psi}_n$.

10.6. *End of the proof of Proposition 10.1.* Let $m \in \mathbb{Z}^*$. In view of Lemma 8.1, we can suppose $m \geq -1$. By continuity of Δ_m, $(\beta_{\boldsymbol{\omega}} \Delta_m \widetilde{\mathcal{V}}^{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \Omega^\bullet,\, \|\boldsymbol{\omega}\| = m}$ is a summable family of $\boldsymbol{A}$, of sum $\Delta_m \widetilde{\psi}_{m+1}$. In particular, the family obtained by extracting the constant terms is summable, but the constant term in $\Delta_m \widetilde{\mathcal{V}}^{\boldsymbol{\omega}}$ is $-V_a^{\boldsymbol{\omega}}(m)$ by (9.10). Hence we get the summability of

$$C_m = \sum_{\|\boldsymbol{\omega}\| = m} \beta_{\boldsymbol{\omega}} V_a^{\boldsymbol{\omega}}(m) \quad \text{in } \mathbb{C},$$

which is the constant term in $-\Delta_m \widetilde{\psi}_{m+1}$. □

10.7. As vector spaces, $\mathbb{C}[[y]]$ and $\boldsymbol{A}[[y]]$ can be identified with $\mathbb{C}^{\mathbb{N}}$ and $\boldsymbol{A}^{\mathbb{N}}$ and are thus also Fréchet spaces if we put the product topology on them.

As an intermediary step in the proof of Theorem 3, let us show

Lemma 10.1. *Let $m \in \mathbb{Z}^*$ and*

$$\mathcal{C}(m) = C_m y^{m+1} \frac{\partial}{\partial y}.$$

Then, for each $n_0 \in \mathbb{N}$, the families $(\widetilde{\mathcal{V}}^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}} y^{n_0})_{\boldsymbol{\omega} \in \Omega^\bullet}$ and $(V_a^{\boldsymbol{\omega}}(m) \boldsymbol{B}_{\boldsymbol{\omega}} y^{n_0})_{\boldsymbol{\omega} \in \Omega^\bullet}$ are summable in $\boldsymbol{A}[[y]]$, of sums $\widetilde{\Theta}^{-1} y^{n_0}$ and $\mathcal{C}(m) y^{n_0}$.

Proof. Our aim is to show that $(\widetilde{\mathcal{V}}^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \Omega^\bullet}$ and $(V_a^{\boldsymbol{\omega}}(m) \boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \Omega^\bullet}$ are pointwise summable families of operators of $\boldsymbol{A}[[y]]$; in view of the above, since $\boldsymbol{B}_{\boldsymbol{\omega}} y = \beta_{\boldsymbol{\omega}} y^{\|\boldsymbol{\omega}\|+1}$, we can already evaluate these operators on y and write

$$\sum_{\boldsymbol{\omega} \in \Omega^\bullet} V_a^{\boldsymbol{\omega}}(m) \boldsymbol{B}_{\boldsymbol{\omega}} y = \sum_{\boldsymbol{\omega} \in \Omega^\bullet,\, \|\boldsymbol{\omega}\| = m} V_a^{\boldsymbol{\omega}}(m) \boldsymbol{B}_{\boldsymbol{\omega}} y = C_m y^{m+1} \quad \text{in } \mathbb{C}[[y]] \tag{10.7}$$

(the first identity stems from (9.11)) and

$$\sum_{\boldsymbol{\omega} \in \Omega^\bullet} \widetilde{\mathcal{V}}^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}} y = y + \sum_{n \geq 0} \widetilde{\psi}_n(z) y^n = \widetilde{\Theta}^{-1} y \quad \text{in } \boldsymbol{A}[[y]].$$

Although similar to formula (3.13), the last equation is stronger in that it gives the sum of a summable family of $\boldsymbol{A}[[y]]$ rather than of a formally summable family of $\mathbb{C}[[z^{-1}, y]]$.

When evaluating the operators $\boldsymbol{B}_{\boldsymbol{\omega}}$ on y^{n_0}, we get coefficients $\beta_{\boldsymbol{\omega}, n_0}$ which generalise the $\beta_{\boldsymbol{\omega}}$'s:

$$\boldsymbol{B}_{\boldsymbol{\omega}} y^{n_0} = \beta_{\boldsymbol{\omega}, n_0} y^{n_0 + \|\boldsymbol{\omega}\|}$$

with $\beta_{\emptyset, n_0} = 1$, $\beta_{(\omega_1), n_0} = n_0$, $\beta_{\boldsymbol{\omega}, n_0} = n_0(n_0 + \check{\omega}_1)(n_0 + \check{\omega}_2) \cdots (n_0 + \check{\omega}_{r-1})$ for $r \geq 2$. Notice that $\beta_{\boldsymbol{\omega}, n_0} \neq 0 \implies \|\boldsymbol{\omega}\| \geq -n_0$.

A suitable modification of the proof of Theorem 2 shows that the families $(\beta_{\boldsymbol{\omega}, n_0} \widehat{\mathcal{V}}^{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \Omega^\bullet,\, \|\boldsymbol{\omega}\| = m}$ are summable in $\hat{\boldsymbol{A}}$ for all $m \geq -n_0$ (replace the functions $\mathcal{S}_m(\zeta) = \frac{m+1}{\zeta - m}$ of Lemma 8.2 by $\frac{m+n_0}{\zeta - m}$, for which the bounds are only slightly worse than in Lemma 8.4).

This yields the first part of the lemma, since we can now write

$$\sum_{\omega\in\Omega^\bullet} \tilde{\mathcal{V}}^{\omega} \boldsymbol{B}_{\omega} y^{n_0} = \sum_{m\geq -n_0} \Big(\sum_{\omega\in\Omega^\bullet,\, \|\omega\|=m} \beta_{\omega,n_0} \tilde{\mathcal{V}}^{\omega} \Big) y^{n_0+m} = \tilde{\Theta}^{-1} y^{n_0} \quad \text{in } \boldsymbol{A}[[y]].$$

By continuity of Δ_m, we also get the summability of $(\beta_{\omega,n_0}\Delta_m \hat{\mathcal{V}}^{\omega})_{\omega\in\Omega^\bullet,\, \|\omega\|=m}$ in $\boldsymbol{A}$, hence of the family $(-\beta_{\omega,n_0} V_a^{\omega}(m))_{\omega\in\Omega^\bullet,\, \|\omega\|=m}$ obtained by extracting the constant terms. Let

$$C_{m,n_0} = \sum_{\omega\in\Omega^\bullet,\, \|\omega\|=m} \beta_{\omega,n_0} V_a^{\omega}(m) \quad \text{in } \mathbb{C}.$$

Thus $(V_a^{\omega}(m)\boldsymbol{B}_{\omega} y^{n_0})_{\omega\in\Omega^\bullet}$ is summable in $\boldsymbol{A}[[y]]$, with sum $C_{m,n_0} y^{n_0+m}$.

Let $\Omega^{k,R}$ $(k, R \in \mathbb{N}^*)$ denote an exhaustion of $\Omega^\bullet$ by finite sets as in the proof of Proposition 6.1. We conclude by showing that $C_{m,n_0} y^{n_0+m} = \mathcal{C}(m) y^{n_0}$. This follows from that fact that the operators

$$\mathcal{C}^{k,R}(m) = \sum_{\omega\in\Omega^{k,R}} V_a^{\omega}(m)\boldsymbol{B}_{\omega}$$

are all derivations of $\mathbb{C}[[y]]$ because of the alternality of $V_a^\bullet(m)$ (the Leibniz rule is easily checked with the help of the cosymmetrality of $\boldsymbol{B}_\bullet$), thus their pointwise limit is also a derivation, which cannot be anything but $\mathcal{C}(m)$ by virtue of (10.7). ☐

10.8. *End of the proof of Theorem 3.* In $\boldsymbol{A}[[y]]$, the families $(\tilde{\mathcal{V}}^{\omega}\boldsymbol{B}_{\omega} y)_{\omega\in\Omega^\bullet}$ and $(\tilde{\mathcal{V}}^{\omega}\boldsymbol{B}_{\omega} y)_{\omega\in\Omega^\bullet}$ are summable, of sums

$$\tilde{\varphi}(z,y) = \sum_{\omega\in\Omega^\bullet} \tilde{\mathcal{V}}^{\omega}(z)\boldsymbol{B}_{\omega} y, \quad \tilde{\psi}(z,y) = \sum_{\omega\in\Omega^\bullet} \tilde{\mathcal{V}}^{\omega}(z)\boldsymbol{B}_{\omega} y. \tag{10.8}$$

The derivation of $\boldsymbol{A}[[y]]$ induced by Δ_m is clearly continuous; applying Δ_m to both sides of the first equation in (10.8) and using (6.1) and (9.10), we find

$$\Delta_m\tilde{\varphi} = \sum_{\omega} (\Delta_m \tilde{\mathcal{V}}^{\omega})\boldsymbol{B}_{\omega} y = \sum_{\omega^1,\omega^2} \tilde{\mathcal{V}}^{\omega^1} V_a^{\omega^2}(m)\boldsymbol{B}_{\omega^2}\boldsymbol{B}_{\omega^1} y$$

$$= \sum_{\omega^2} V_a^{\omega^2}(m)\boldsymbol{B}_{\omega^2}\tilde{\Theta} y = \mathcal{C}(m)\tilde{\varphi}$$

(with the help of Lemma 10.1 for the last identities). Similarly,

$$\Delta_m\tilde{\psi} = \sum_{\omega} (\Delta_m \tilde{\mathcal{V}}^{\omega})\boldsymbol{B}_{\omega} y = -\sum_{\omega^1,\omega^2} V_a^{\omega^1}(m)\tilde{\mathcal{V}}^{\omega^2}\boldsymbol{B}_{\omega^2}\boldsymbol{B}_{\omega^1} y$$

$$= -\sum_{\omega^2} \tilde{\mathcal{V}}^{\omega^2}\boldsymbol{B}_{\omega^2}\mathcal{C}(m) y = -\tilde{\Theta}^{-1}(C_m y^{m+1}) = -C_m(\tilde{\Theta}^{-1} y)^{m+1}. \quad \square$$

10.9. *Operator form of the Bridge Equation.* As announced after the statement of Theorem 3, the Bridge Equation can be given a form which involves the operators $\tilde{\Theta}$ or $\tilde{\Theta}^{-1}$ in a more symmetric way. This will require a further construction.

Proposition 10.2. *Let* $\boldsymbol{A} = \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$. *The set*

$$\boldsymbol{A}\{y\} = \Big\{ \textstyle\sum_{n\ge 0} \tilde f_n(z) y^n \in \boldsymbol{A}[[y]] \mid \text{for all } K \in \mathcal{K} \text{ there exist } c, \Lambda > 0 \\ \text{such that } \|\tilde f_n\|_K \le c\Lambda^n \text{ for all } n \Big\}$$

is a subalgebra of $\boldsymbol{A}[[y]]$, *which contains* $\widetilde\varphi(z,y)$ *and* $\widetilde\psi(z,y)$ *and which is invariant by all the alien derivations* Δ_m. *Moreover, the substitution operators* $\widetilde\Theta$ *and* $\widetilde\Theta^{-1}$ *leave* $\boldsymbol{A}\{y\}$ *invariant and the operators they induce on* $\boldsymbol{A}\{y\}$ *satisfy the "alien chain rule"*

$$\Delta_m \widetilde\Theta \tilde f = \widetilde\Theta \Delta_m \tilde f + (\widetilde\Theta \partial_y \tilde f)\Delta_m \widetilde\varphi, \quad \Delta_m \widetilde\Theta^{-1} \tilde f = \widetilde\Theta^{-1} \Delta_m \tilde f + (\widetilde\Theta^{-1} \partial_y \tilde f)\Delta_m \widetilde\psi.$$

Idea of the proof. The fact that $\widetilde\varphi, \widetilde\psi \in \boldsymbol{A}\{y\}$ follows easily from (8.3)–(8.4). The other statements require symmetrically contractile paths, first to control the seminorm $\|\,.\,\|_K$ of a product of simple resurgent functions ($\boldsymbol{A}$ is in fact a Fréchet algebra), and then to study $\partial_y^n \tilde f(z, \widetilde\varphi_0(z))$ which appears in the substitution of $\widetilde\varphi$ inside a series with resurgent coefficients:

$$\tilde f(z,\widetilde\varphi) = \tilde f(z,\widetilde\varphi_0) + y\partial_y \tilde f(z,\widetilde\varphi_0)\widetilde\Phi_1 + y^2\Big(\partial_y \tilde f(z,\widetilde\varphi_0)\widetilde\Phi_2 + \frac{1}{2!}\partial_y^2 \tilde f(z,\widetilde\varphi_0)\widetilde\Phi_1^2\Big) + \cdots$$

with the notation (10.6). See [14] (e.g. §2.3, formula (41)). □

Theorem 4. *We have the following identities in* $\mathrm{End}_{\mathbb{C}}(\boldsymbol{A}\{y\})$:

$$\big[\Delta_m, \widetilde\Theta\big] = \mathcal{C}(m)\widetilde\Theta, \quad \big[\Delta_m, \widetilde\Theta^{-1}\big] = -\widetilde\Theta^{-1}\mathcal{C}(m), \tag{10.9}$$

for all $m \in \mathbb{Z}^*$.

Proof. We must prove that $\widetilde\Theta \Delta_m \widetilde\Theta^{-1} - \Delta_m = -\mathcal{C}(m)$.

The operators $\widetilde\Theta$ and $\widetilde\Theta^{-1}$ are mutually inverse $\boldsymbol{A}$-linear automorphisms of $\mathcal{A} = \boldsymbol{A}\{y\}$ and $\mathcal{C}(m)$ is an $\boldsymbol{A}$-linear derivation. The operator Δ_m is a derivation, it is not $\boldsymbol{A}$-linear, but $D = \widetilde\Theta \Delta_m \widetilde\Theta^{-1} - \Delta_m$ is an $\boldsymbol{A}$-linear derivation; indeed, if $\mu(z) \in \boldsymbol{A}$ and $f(z,y) \in \mathcal{A}$, then

$$\begin{aligned} D(\mu f) &= \widetilde\Theta \Delta_m (\mu \widetilde\Theta^{-1} f) - \Delta_m(\mu f) \\ &= \widetilde\Theta\big(\mu \Delta_m \widetilde\Theta^{-1} f + (\Delta_m \mu)\widetilde\Theta^{-1} f\big) - \big(\mu \Delta_m f + (\Delta_m \mu) f\big) \\ &= \mu \widetilde\Theta \Delta_m \widetilde\Theta^{-1} f + (\Delta_m \mu) f - \mu \Delta_m f - (\Delta_m \mu) f = \mu D f. \end{aligned}$$

It is thus sufficient to check that the operator $D + \mathcal{C}(m)$ vanishes on y (being a continuous $\boldsymbol{A}$-linear derivation of $\mathcal{A}$, it will have to vanish everywhere).

But, in view of (10.5), $Dy = \widetilde\Theta \Delta_m \widetilde\psi = -C_m(\widetilde\Theta\widetilde\psi)^{m+1} = -C_m y^{m+1}$, as required. □

10.10. *The Bridge Equation and the problem of analytic classification.* We now explain why the coefficients C_m implied in the Bridge Equation are "analytic invariants" of the vector field X.

Suppose we are given two saddle-node vector fields, X_1 and X_2, of the form (2.1) and satisfying (2.2). Both of them are formally conjugate to the normal form X_0, hence they are mutually formally conjugate. Namely, we have formal substitution automorphisms Θ_i (or $\widetilde{\Theta}_i$, when using the variable z instead of x) conjugating X_i with X_0, for $i = 1, 2$, hence

$$\Theta X_1 = X_2 \Theta, \quad \Theta = \Theta_2^{-1}\Theta_1.$$

The operator Θ is the substitution operator associated with

$$\theta\colon (x, y) \mapsto \big(x, \varphi(x, y)\big), \quad \varphi(x, y) = \Theta y = \varphi_1\big(x, \psi_2(x, y)\big),$$

which is the unique formal transformation of the form (2.5) such that $X_1 = \theta_* X_2$.

One can check that, when passing to the variable z, one gets as a consequence of Proposition 10.2 and Theorem 4:

$$\widetilde{\Theta} \in \operatorname{End}_{\mathbb{C}}(\boldsymbol{A}\{y\}), \quad \big[\Delta_m, \widetilde{\Theta}\big] = \widetilde{\Theta}_2^{-1}\big(\mathcal{C}_1(m) - \mathcal{C}_2(m)\big)\widetilde{\Theta}_1, \quad m \in \mathbb{Z}^*,$$

where $\mathcal{C}_i(m) = C_{i,m} y^{m+1} \frac{\partial}{\partial y}$ is the derivation appearing in the right-hand side of the Bridge Equations (10.9) for X_i.

If X_1 and X_2 are holomorphically conjugate, then the unique formal conjugacy θ is given by a convergent series $\theta(x, y)$, thus all the alien derivatives of $\widetilde{\varphi}$ vanish and $\mathcal{C}_1(m) = \mathcal{C}_2(m)$ for all m. We thus have proved half of

Theorem 5. *Two saddle-node vector fields of the form* (2.1) *and satisfying* (2.2) *are analytically conjugate if and only if their Bridge Equations* (10.3) *share the same collection of coefficients* $(C_m)_{m\in\mathbb{Z}^*}$.

According to this theorem, the numbers C_m thus constitute a complete system of analytic invariants for a saddle-node vector field.

To complete the proof of Theorem 5, one needs to show the reverse implication, i.e. that the identities $\mathcal{C}_1(m) = \mathcal{C}_2(m)$ imply the convergence of $\varphi_1\big(x, \psi_2(x, y)\big)$. This will follow from the results of next section, according to which the coefficients C_m are related to another complete system of analytic invariants, which admits a more geometric description.

10.11. We end this section with a look at simple cases of the general theory.

"Euler equation" corresponds to $A(x, y) = x + y$, as mentioned in Section 2. We may call Euler-like equations those which correspond to the case in which $a_\eta = 0$ for $\eta \geq 1$, thus $A(x, y) = a_{-1}(x) + \big(1 + a_0(x)\big)y$. For them, the formal integral is explicit.

Set $\widetilde{a}_0(z) = a_0(-1/z) \in z^{-2}\mathbb{C}\{z^{-1}\}$ and $\widetilde{a}_{-1}(z) = a_{-1}(-1/z) \in z^{-1}\mathbb{C}\{z^{-1}\}$ as usual. Let $\widetilde{\alpha}(z)$ be the unique series such that $\partial_z\widetilde{\alpha} = \widetilde{a}_0$ and $\widetilde{\alpha} \in z^{-1}\mathbb{C}\{z^{-1}\}$. Set also $\widetilde{\beta} = \widetilde{a}_{-1}\,\mathrm{e}^{-\widetilde{\alpha}} \in z^{-1}\mathbb{C}\{z^{-1}\}$ and $\widehat{\beta} = \mathcal{B}\widetilde{\beta}$ (which is an entire function of exponential type). One finds

$$\widetilde{Y}(z,u) = \widetilde{\varphi}_0(z) + u\, \mathrm{e}^{z+\widetilde{\alpha}(z)}, \quad \widetilde{\varphi}_0 = -\mathrm{e}^{\widetilde{\alpha}}\, \mathcal{B}^{-1}\Big(\zeta \mapsto \frac{\widehat{\beta}(\zeta)}{\zeta+1}\Big).$$

Correspondingly, $\varphi(x,y) = \Phi_0(x) + \Phi_1(x)y$ with $\Phi_0(x) = \widetilde{\varphi}_0(-1/x)$ generically divergent and $\Phi_1(x) = \mathrm{e}^{\widetilde{\alpha}(-1/x)}$ convergent.

One has $C_m = 0$ for every $m \in \mathbb{Z} \setminus \{-1\}$, but

$$C_{-1} = \mathrm{e}^{-\widetilde{\alpha}} \Delta_{-1}\widetilde{\varphi}_0 = -2\pi\mathrm{i}\, \widehat{\beta}(-1).$$

10.12. Another particular case, much less trivial, is that of Riccati equations (see [4], [3, Vol. 2] or [1] for the resurgent approach, or [12]): when $a_1 \neq 0$ and $a_\eta = 0$ for $\eta \geq 2$, hence $A(x,y) = a_{-1}(x) + \big(1 + a_0(x)\big)y + a_1(x)y^2$, one can check that the formal integral has a linear fractional dependence upon the parameter u:

$$\widetilde{Y}(z,u) = \frac{\widetilde{\varphi}_0(z) + u\, \mathrm{e}^z \widetilde{\chi}(z)}{1 + u\, \mathrm{e}^z \widetilde{\chi}(z) \widetilde{\varphi}_\infty(z)},$$

where $\widetilde{\varphi}_0$, $\widetilde{\varphi}_\infty$ and $-1 + \widetilde{\chi}$ belong to $z^{-1}\mathbb{C}[[z^{-1}]]$; $\widetilde{\varphi}_0$ and $1/\widetilde{\varphi}_\infty$ can be found as the unique solutions of the differential equation (2.7) in the fraction field $\mathbb{C}((z^{-1}))$. Correspondingly, the normalising series $\varphi(x,y)$ and $\psi(x,y)$ have a linear fractional dependence upon y.

In the Riccati case, only C_{-1} and C_1 may be nonzero. Indeed,

$$\Delta_m\widetilde{\varphi}_0 \neq 0 \implies m = -1, \quad \Delta_m\widetilde{\varphi}_\infty \neq 0 \implies m = 1, \quad \Delta_m\widetilde{\chi} \neq 0 \implies m = \pm 1.$$

10.13. We may call "canonical Riccati equations" the equations corresponding to a function A of the form $A(x,y) = y + \frac{1}{2\pi\mathrm{i}} B_- x + \frac{1}{2\pi\mathrm{i}} B_+ x y^2$, with $B_-, B_+ \in \mathbb{C}$. Thus, for them, the differential equation (2.7) reads

$$\partial_z \widetilde{Y} = \widetilde{Y} - \frac{1}{2\pi\mathrm{i}z}(B_- + B_+\widetilde{Y}^2).$$

A direct mould computation based on (10.1) is given in [3], Vol. 2, pp. 476–480, yielding

$$C_{-1} = B_-\sigma(B_-B_+), \quad C_1 = -B_+\sigma(B_-B_+),$$

with $\sigma(b) = \frac{2}{b^{1/2}} \sin \frac{b^{1/2}}{2}$. See [12] (or [1]) for a computation by another method.

11 Relation with Martinet–Ramis's invariants

In this section, we continue to investigate the consequences of the resurgence of the solution of the conjugacy equation for a saddle-node X. We shall now connect the "alien computations" of the previous section with Martinet–Ramis's solution of the problem of analytic classification [12], completing at the same time the proof of Theorem 5.

This will be done by comparing sectorial solutions of the conjugacy problem obtained by Borel–Laplace summation on the one hand, and by deriving geometric consequences of the Bridge Equation through exponentiation and summation on the other hand (this amounts to a resurgent description of the "Stokes phenomenon" for the differential equation (2.7)).

11.1. Let us call *Martinet–Ramis's invariants* of X the numbers $\xi_{-1}, \xi_1, \xi_2, \ldots$ defined in terms of Écalle's invariants by the formulas

$$\xi_{-1} = -C_{-1}, \tag{11.1}$$

$$\xi_m = \sum_{r \geq 1} \sum_{\substack{m_1,\ldots,m_r \geq 1 \\ m_1+\cdots+m_r=m}} \frac{(-1)^r}{r!} \beta_{m_1,\ldots,m_r} C_{m_1} \ldots C_{m_r}, \quad m \geq 1, \tag{11.2}$$

where, as usual, $\beta_{m_1} = 1$ and $\beta_{m_1,\ldots,m_r} = (m_1+1)(m_1+m_2+1)\cdots(m_1+\cdots+m_{r-1})$ for $r \geq 2$.

Observe that they are obtained by integrating backwards the vector fields

$$\mathcal{C}_- = \mathcal{C}(-1) = C_{-1}\frac{\partial}{\partial u}, \quad \mathcal{C}_+ = \sum_{m>0} \mathcal{C}(m) = \sum_{m>0} C_m u^{m+1}\frac{\partial}{\partial u}.$$

Indeed, the time-(-1) maps of $\mathcal{C}_-$ and $\mathcal{C}_+$ are

$$u \mapsto \xi_-(u) = u + \xi_{-1}, \quad u \mapsto \xi_+(u) = u + \sum_{m>0} \xi_m u^{m+1} \tag{11.3}$$

(as can be checked by viewing $-\mathcal{C}_+$ as an elementary mould-comould expansion on the alphabet $\mathbb{N}^*$; the reason for changing the variable y into u will appear later).[14]

These numbers can also be defined directly from the iterated integrals $L_a^{\boldsymbol{\omega}}$ of (9.18):

Proposition 11.1. *The family* $\left(\beta_{\boldsymbol{\omega}} L_a^{\boldsymbol{\omega}}\right)_{\boldsymbol{\omega}\in\Omega^\bullet,\, \|\boldsymbol{\omega}\|=m}$ *is summable in* $\mathbb{C}$ *for each* $m \in \mathbb{Z}^*$ *and*

$$\xi_m = \sum_{\boldsymbol{\omega}\in\Omega^\bullet,\, \|\boldsymbol{\omega}\|=m} \beta_{\boldsymbol{\omega}} L_a^{\boldsymbol{\omega}},$$

with the convention $\xi_m = 0$ *for* $m \leq -2$.

Idea of the proof. The relations $\xi_\pm(u) = \sum_{\boldsymbol{\omega}\in\Omega^\bullet} L_{a,\pm}^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}} u$ (where $L_{a,\pm}^\bullet$ is defined by (9.22)) formally follow from the formula $L_{a,\pm}^\bullet = \exp\left(-V_{a,\pm}^\bullet\right)$ and Lemma 10.1, according to which $(V_{a,\pm}^{\boldsymbol{\omega}} \boldsymbol{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\Omega^\bullet}$ is a pointwise summable family of operators of $\boldsymbol{A}[[u]]$ with sum $\mathcal{C}_\pm$. The summability can be justified by the same kind of arguments as in the proof of Proposition 10.1 and Theorem 3. □

11.2. The formulas (11.1)–(11.2) can be inverted so as to express the C_m's in terms of the ξ_m's. Theorem 5 is thus equivalent to the fact that the ξ_m's constitute themselves

[14] Thus one always has $\xi_-(u) = u - C_{-1}$, and in the Riccati case as at the end of the previous section $\xi_+(u) = \frac{u}{1-C_1 u}$.

a complete system of analytic invariants for the saddle-node classification problem. We shall now prove this fact directly.

In fact, we shall obtain more: the pair (ξ_-, ξ_+) is a complete system of analytic invariants and ξ_+ *is necessarily convergent*. Thus, not all collections of numbers $(C_m)_{m\in\{-1\}\cup\mathbb{N}^*}$ can appear as analytic invariants, only those for which the corresponding ξ_m's admit geometric bounds $|\xi_m| \le K^m$ for $m \ge 1$ (hence they have to satisfy Gevrey bounds themselves: $|C_m| \le K_1^m m!$ for $m \ge 1$).

This information will follow from the geometric interpretation of $\xi_\pm$. Martinet and Ramis have also showed that any collection $(\xi_m)_{m\in\{-1\}\cup\mathbb{N}^*}$ subject to the previous growth constraint can be obtained as a system of analytic invariants for some saddle-node vector field, but we shall not consider this question here.

11.3. Let us consider the saddle-node vector field X and its normal form X_0 in the variable $z = -1/x$ instead of x:

$$\widetilde{X} = \frac{\partial}{\partial z} + A(-1/z, y)\frac{\partial}{\partial y}, \quad \widetilde{X}_0 = \frac{\partial}{\partial z} + y\frac{\partial}{\partial y}.$$

For $\varepsilon \in \,]0, \pi/2[$ and $R > 0$, we set

$$\mathcal{D}^{\mathrm{up}}(R, \varepsilon) = \{\, z \in \mathbb{C} \mid -\tfrac{\pi}{2} + \varepsilon \le \arg z \le \tfrac{3\pi}{2} - \varepsilon,\ |z| \ge R \,\},$$
$$\mathcal{D}^{\mathrm{low}}(R, \varepsilon) = \{\, z \in \mathbb{C} \mid -\tfrac{3\pi}{2} + \varepsilon \le \arg z \le \tfrac{\pi}{2} - \varepsilon,\ |z| \ge R \,\},$$

which are "sectorial neighbourhoods of infinity" in the z-plane (corresponding to certain sectorial neighbourhoods of the origin in the x-plane). Their intersection has two connected components:

$$\mathcal{D}_-(R, \varepsilon) = \{\, z \in \mathbb{C} \mid \tfrac{\pi}{2} + \varepsilon \le \arg z \le \tfrac{3\pi}{2} - \varepsilon,\ |z| \ge R \,\} \subset \{\, \Re\mathrm{e}\, z < 0 \,\},$$
$$\mathcal{D}_+(R, \varepsilon) = \{\, z \in \mathbb{C} \mid -\tfrac{\pi}{2} + \varepsilon \le \arg z \le \tfrac{\pi}{2} - \varepsilon,\ |z| \ge R \,\} \subset \{\, \Re\mathrm{e}\, z > 0 \,\}.$$

Theorem 6. *Let $\varepsilon \in \,]0, \pi/2[$. Then there exist $R, \rho > 0$ such that:*

(i) *By Borel–Laplace summation, the formal series $\widetilde{\varphi}_n(z)$ give rise to functions $\widetilde{\varphi}_n^{\mathrm{up}}(z)$, resp. $\widetilde{\varphi}_n^{\mathrm{low}}(z)$, which are analytic in $\mathcal{D}^{\mathrm{up}}(R, \varepsilon)$, resp. $\mathcal{D}^{\mathrm{low}}(R, \varepsilon)$, such that the formulas*

$$\widetilde{\varphi}^{\mathrm{up}}(z, y) = y + \sum_{n\ge 0} \widetilde{\varphi}_n^{\mathrm{up}}(z) y^n, \quad \widetilde{\varphi}^{\mathrm{low}}(z, y) = y + \sum_{n\ge 0} \widetilde{\varphi}_n^{\mathrm{low}}(z) y^n$$

define two functions $\widetilde{\varphi}^{\mathrm{up}}$ and $\widetilde{\varphi}^{\mathrm{low}}$ analytic in $\mathcal{D}^{\mathrm{up}}(R, \varepsilon) \times \{\, |y| \le \rho \,\}$, resp. $\mathcal{D}^{\mathrm{low}}(R, \varepsilon) \times \{\, |y| \le \rho \,\}$, and each of the transformations

$$\widetilde{\theta}^{\mathrm{up}}(z, y) = \big(z, \widetilde{\varphi}^{\mathrm{up}}(z, y)\big), \quad \widetilde{\theta}^{\mathrm{low}}(z, y) = \big(z, \widetilde{\varphi}^{\mathrm{low}}(z, y)\big)$$

is injective in its domain and establishes there a conjugacy between the normal form $\widetilde{X}_0$ and the saddle-node vector field $\widetilde{X}$.

(ii) *The series* ξ_+ *of* (11.3) *has positive radius of convergence and the upper and lower normalisations are connected by the formulas*

$$\widetilde{\theta}^{\mathrm{up}}(z,y) = \widetilde{\theta}^{\mathrm{low}}\big(z, \xi_-(y\,\mathrm{e}^{-z})\,\mathrm{e}^{z}\big) = \widetilde{\theta}^{\mathrm{low}}(z, y + \xi_{-1}\mathrm{e}^{z})$$

for $z \in \mathcal{D}_-(R,\varepsilon)$ *and* $|y| \le \rho$*, whereas*

$$\widetilde{\theta}^{\mathrm{low}}(z,y) = \widetilde{\theta}^{\mathrm{up}}\big(z, \xi_+(y\,\mathrm{e}^{-z})\,\mathrm{e}^{z}\big) = \widetilde{\theta}^{\mathrm{up}}\big(z, y + \xi_1 y^2\mathrm{e}^{-z} + \xi_2 y^3\mathrm{e}^{-2z} + \cdots\big)$$

for $z \in \mathcal{D}_+(R,\varepsilon)$ *and* $|y| \le \rho$.

(iii) *The pair* (ξ_-, ξ_+) *is a complete system of analytic invariants for* X.

As already mentioned, Theorem 6 contains Theorem 5. The rest of this section is devoted to the proof of Theorem 6.

Remark 11.1. For the Borel–Laplace sums of the formal integral, this yields $\widetilde{Y}^{\mathrm{up}}(z,u) = \widetilde{Y}^{\mathrm{low}}\big(z,\xi_-(u)\big)$ if $z \in \mathcal{D}_-(R,\varepsilon)$ and $\widetilde{Y}^{\mathrm{low}}(z,u) = \widetilde{Y}^{\mathrm{up}}\big(z,\xi_+(u)\big)$ if $z \in \mathcal{D}_+(R,\varepsilon)$.

11.4. In view of inequalities (8.4)–(8.5), the principal branches of the Borel transforms $\widehat{\varphi}_n(\zeta)$ and $\widehat{\psi}_n(\zeta)$ admit exponential bounds of the form $KL^n\,\mathrm{e}^{C|\zeta|}$ in the sectors $\{\,\zeta \in \mathbb{C} \mid \frac{\varepsilon}{2} \le \arg\zeta \le \pi - \frac{\varepsilon}{2}\,\}$ and $\{\,\zeta \in \mathbb{C} \mid \pi + \frac{\varepsilon}{2} \le \arg\zeta \le 2\pi - \frac{\varepsilon}{2}\,\}$. Using the directions of the first sector for instance, we can define analytic functions by gluing the Laplace transforms corresponding to various directions

$$\widetilde{\varphi}_n^{\mathrm{low}}(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \widehat{\varphi}_n(\zeta)\,\mathrm{e}^{-z\zeta}\,\mathrm{d}\zeta, \quad \widetilde{\psi}_n^{\mathrm{low}}(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \widehat{\psi}_n(\zeta)\,\mathrm{e}^{-z\zeta}\,\mathrm{d}\zeta,$$

with $\theta \in [\frac{\varepsilon}{2}, \pi - \frac{\varepsilon}{2}]$. If we take R large enough, then the union of the half-planes $\{\,\Re\mathrm{e}(z\,\mathrm{e}^{\mathrm{i}\theta}) > C\,\}$ contains $\mathcal{D}^{\mathrm{low}}(R,\varepsilon)$ and the functions

$$\widetilde{\varphi}^{\mathrm{low}}(z,y) = y + \sum_{n\ge 0} \widetilde{\varphi}_n^{\mathrm{low}}(z)y^n, \quad \widetilde{\psi}^{\mathrm{low}}(z,y) = y + \sum_{n\ge 0} \widetilde{\psi}_n^{\mathrm{low}}(z)y^n$$

are analytic for $z \in \mathcal{D}^{\mathrm{low}}(R,\varepsilon)$ and $|y| \le \rho$ as soon as $\rho < 1/L$.

The standard properties of Borel–Laplace summation ensure that the relations $y = \widetilde{\varphi}\big(z, \widetilde{\psi}(z,y)\big) = \widetilde{\psi}\big(z, \widetilde{\varphi}(z,y)\big)$ and $\widetilde{X}_0\widetilde{\varphi}(z,y) = A\big(-1/z, \widetilde{\varphi}(z,y)\big)$ yield similar relations for $\widetilde{\varphi}^{\mathrm{low}}$ and $\widetilde{\psi}^{\mathrm{low}}$, possibly in smaller domains (because $\widetilde{\varphi}^{\mathrm{low}}(z,y) - y$ and $\widetilde{\psi}^{\mathrm{low}}(z,y) - y$ can be made uniformly small by increasing R and diminishing ρ). Hence the transformations

$$(z,y) \mapsto \big(z, \widetilde{\varphi}^{\mathrm{low}}(z,y)\big), \quad (z,y) \mapsto \big(z, \widetilde{\psi}^{\mathrm{low}}(z,y)\big)$$

(or rather the sectorial germs they represent) are mutually inverse and establish a conjugacy between $\widetilde{X}_0$ and $\widetilde{X}$.

We define similarly $\widetilde{\varphi}^{\mathrm{up}}(z,y)$ and $\widetilde{\psi}^{\mathrm{up}}(z,y)$ with the desired properties, by means of Laplace transforms in directions belonging to $[\pi + \frac{\varepsilon}{2}, 2\pi - \frac{\varepsilon}{2}]$. This yields the first statement in Theorem 6.

11.5. We now have at our disposal two sectorial normalisations

$$\widetilde{\theta}^{\text{low}}\colon (z,y)\mapsto \big(z,\widetilde{\varphi}^{\text{low}}(z,y)\big),\quad \widetilde{\theta}^{\text{up}}\colon (z,y)\mapsto \big(z,\widetilde{\varphi}^{\text{up}}(z,y)\big),$$

which are defined in different but overlapping domains, and which admit the same asymptotic expansion with respect to z (when one first expands in powers of y). If we consider $\big(\widetilde{\theta}^{\text{up}}\big)^{-1}\circ\widetilde{\theta}^{\text{low}}$ or $\big(\widetilde{\theta}^{\text{low}}\big)^{-1}\circ\widetilde{\theta}^{\text{up}}$ in one of the two components of $\mathcal{D}^{\text{low}}(R,\varepsilon)\cap\mathcal{D}^{\text{up}}(R,\varepsilon)$, we thus get a transformation of the form

$$(z,y)\mapsto\big(z,\chi(z,y)\big) \tag{11.4}$$

which conjugates the normal form $\widetilde{X}_0$ with itself, to which one can apply the following:

Lemma 11.1. *Let $\mathcal{D}$ be a domain in $\mathbb{C}$. Suppose that the transformation $(z,y)\mapsto \big(z,\chi(z,y)\big)$ is analytic and injective for $z\in\mathcal{D}$ and $|y|\le\rho$, and that it conjugates $\widetilde{X}_0$ with itself. Then there exists $\xi(u)\in\mathbb{C}\{u\}$ such that*

$$\chi(z,y)=\xi(y\,\mathrm{e}^{-z})\mathrm{e}^{z}. \tag{11.5}$$

Such transformations are called *sectorial isotropies* of the normal form.

Proof. By assumption $\chi=\widetilde{X}_0\chi$. Since $y=\widetilde{X}_0 y$, this implies that $\frac{1}{y}\chi(z,y)$ is a first integral of $\widetilde{X}_0$. Thus $\frac{1}{u\,\mathrm{e}^{z}}\chi(z,u\,\mathrm{e}^{z})$ is independent of z and can be written $\frac{\xi(u)}{u}$, where obviously $\xi(u)\in\mathbb{C}\{u\}$. □

When $\chi(z,y)$ comes from $\big(\widetilde{\theta}^{\text{up}}\big)^{-1}\circ\widetilde{\theta}^{\text{low}}$ or $\big(\widetilde{\theta}^{\text{low}}\big)^{-1}\circ\widetilde{\theta}^{\text{up}}$, we have a further piece of information: in the Taylor expansion $\chi(z,y)-y=\sum_{n\ge0}\chi_n(z)y^n$, each component $\chi_n(z)$ admits the null series as asymptotic expansion in $\mathcal{D}_\pm(R,\varepsilon)$ (the transformation (11.4) is asymptotic to the identity because $\widetilde{\theta}^{\text{up}}$ and $\widetilde{\theta}^{\text{low}}$ share the same asymptotic expansion). This has different implications according to whether the domain $\mathcal{D}$ is $\mathcal{D}_-(R,\varepsilon)$ or $\mathcal{D}_+(R,\varepsilon)$.

Indeed, if we expand $\xi(u)-u=\sum_{n\ge0}\alpha_n u^n$, we get

$$\chi_0(z)=\alpha_0\,\mathrm{e}^{z},\quad \chi_1(z)=\alpha_1,\quad \chi_2(z)=\alpha_2\,\mathrm{e}^{-z},\quad \chi_3(z)=\alpha_3\,\mathrm{e}^{-2z},\ \ldots$$

hence

$$\begin{aligned}
\mathcal{D}=\mathcal{D}_-(R,\varepsilon)\subset\{\,\Re e\, z<0\,\} &\implies \alpha_n=0\quad\text{for } n\ne0,\\
\mathcal{D}=\mathcal{D}_+(R,\varepsilon)\subset\{\,\Re e\, z>0\,\} &\implies \alpha_0=\alpha_1=0.
\end{aligned}$$

The upshot is that there exist $\alpha_0\in\mathbb{C}$ and $\xi(u)=u+\alpha_2u^2+\alpha_3u^3+\ldots\in\mathbb{C}\{u\}$ such that

$$\begin{aligned}
&\big(\widetilde{\theta}^{\text{low}}\big)^{-1}\circ\widetilde{\theta}^{\text{up}}(z,y)=(z,y+\alpha_0\mathrm{e}^{z}), && z\in\mathcal{D}_-(R,\varepsilon),\\
&\big(\widetilde{\theta}^{\text{up}}\big)^{-1}\circ\widetilde{\theta}^{\text{low}}(z,y)=(z,y+\alpha_2y^2\mathrm{e}^{-z}+\alpha_3y^3\mathrm{e}^{-2z}+\cdots), && z\in\mathcal{D}_+(R,\varepsilon).
\end{aligned}$$

11.6. It is elementary to check that the pair of sectorial isotropies

$$\Big(\big(\widetilde\theta^{\mathrm{low}}\big)^{-1} \circ \widetilde\theta^{\mathrm{up}}{}_{|\mathcal{D}_-(R,\varepsilon)}, \big(\widetilde\theta^{\mathrm{up}}\big)^{-1} \circ \widetilde\theta^{\mathrm{low}}{}_{|\mathcal{D}_+(R,\varepsilon)} \Big)$$

is a complete system of analytic invariants for X: suppose indeed that two saddle-node vector fields X_1 and X_2 are given and that we wish to know whether the unique formal transformation θ of the form (2.5) which conjugate them is convergent, then $\widetilde\theta_2^{\mathrm{up}} \circ \big(\widetilde\theta_1^{\mathrm{up}}\big)^{-1}$ and $\widetilde\theta_2^{\mathrm{low}} \circ \big(\widetilde\theta_1^{\mathrm{low}}\big)^{-1}$ are two sectorial conjugacies between X_1 and X_2 defined in different but overlapping domains and admitting θ as asymptotic expansion (up to the change $x = -1/z$); they coincide and define an analytic conjugacy iff $\big(\widetilde\theta_2^{\mathrm{low}}\big)^{-1} \circ \widetilde\theta_2^{\mathrm{up}} = \big(\widetilde\theta_1^{\mathrm{low}}\big)^{-1} \circ \widetilde\theta_1^{\mathrm{up}}$ in both components of the intersection of the domains.

11.7. Therefore, it only remains to be checked that $\alpha_0 = \xi_1$ and $\xi = \xi_+$. This will follow from the interpretation of the operators Δ_m^+ as components of the "Stokes automorphism". For this part, the reader may consult the end of §2.4 in [14].

Suppose that a simple resurgent functions $c\,\delta + \widehat\varphi \in \widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}$ has the following property: the functions $\widehat\chi_m$ defined by $\Delta_m^+(c\,\delta + \widehat\varphi) = \gamma_m\,\delta + \widehat\chi_m$ and $\widehat\varphi$ itself have at most exponential growth in each non-horizontal directions, so that one can consider the Laplace transforms $\mathcal{L}^\theta\widehat\varphi(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \widehat\varphi(\zeta)\,\mathrm{e}^{-z\zeta}\,\mathrm{d}\zeta$ or $\mathcal{L}^\theta\widehat\chi_m(z)$ for $\theta \in \,]\varepsilon, \pi-\varepsilon[$ or $\theta \in \,]\pi+\varepsilon, 2\pi-\varepsilon[$, which are analytic in sectorial neighbourhoods of infinity of the form $\mathcal{D}^{\mathrm{low}}(R,\varepsilon)$ or $\mathcal{D}^{\mathrm{up}}(R,\varepsilon)$. Let $\theta < 0 < \theta'$, with θ and θ' both close to 0; by deforming a contour of integration, one deduces from the definition (9.20) that, for any $M \in \mathbb{N}^*$ and $\sigma \in \,]0,1[$,

$$c + \mathcal{L}^\theta\widehat\varphi(z) = c + \mathcal{L}^{\theta'}\widehat\varphi(z) + \sum_{m=1}^{M} \mathrm{e}^{-mz}\big(\gamma_m + \mathcal{L}^{\theta'}\widehat\chi_m(z)\big) + O(|\mathrm{e}^{-(M+\sigma)z}|)$$

in the sectorial neighbourhood of infinity obtained by imposing that both $\Re\mathrm{e}(z\,\mathrm{e}^{\mathrm{i}\theta})$ and $\Re\mathrm{e}(z\,\mathrm{e}^{\mathrm{i}\theta'})$ be large enough, which is contained in the right half-plane $\{\,\Re\mathrm{e}\, z > 0\,\}$.

Let us denote this by: $\mathcal{L}^\theta(c\,\delta+\widehat\varphi) \sim \sum_{m\ge0} \mathrm{e}^{-mz}\mathcal{L}^{\theta'}\Delta_m^+(c\,\delta+\widehat\varphi)$ in $\{\,\Re\mathrm{e}\, z > 0\,\}$. Similarly, if $\theta < \pi < \theta'$ with θ and θ' both close to π, one gets $\mathcal{L}^\theta(c\,\delta + \widehat\varphi) \sim \sum_{m\le0} \mathrm{e}^{-mz}\mathcal{L}^{\theta'}\Delta_m^+(c\,\delta + \widehat\varphi)$ in the left half-plane $\{\,\Re\mathrm{e}\, z < 0\,\}$.

We can even write $\mathcal{L}^\theta \sim \mathcal{L}^{\theta'} \circ \sum_{m\ge0} \dot\Delta_m^+$ in $\{\,\Re\mathrm{e}\, z > 0\,\}$ and $\mathcal{L}^\theta \sim \mathcal{L}^{\theta'} \circ \sum_{m\le0} \dot\Delta_m^+$ in $\{\,\Re\mathrm{e}\, z < 0\,\}$, if we define properly $\dot\Delta_m^+$ in the convolutive model. See [14]: $\dot\Delta_m^+ = \tau_m \circ \Delta_m^+$, with a shift operator $\tau_m\colon \widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}} \to \tau_m(\widehat{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}})$, the target space being the set of simple resurgent functions "based at m" (instead of being based at the origin). On the other hand, we can rephrase (9.25) as $\sum_{m\ge0} \dot\Delta_m^+ = \exp\big(\sum_{m>0} \dot\Delta_m\big)$, $\sum_{m\le0} \dot\Delta_m^+ = \exp\big(\sum_{m<0} \dot\Delta_m\big)$.

Apply this to $\widetilde Y(z,u)$ (or, rather, to each of its components): when θ and θ' are close to 0, we have $\mathcal{L}^\theta\widehat Y = \widetilde Y^{\mathrm{up}}$ and $\mathcal{L}^{\theta'}\widehat Y = \widetilde Y^{\mathrm{low}}$ in $\mathcal{D}_+(R,\varepsilon)$, hence, in view of the Bridge Equation, $\widetilde Y^{\mathrm{up}} \sim (\mathcal{L}^{\theta'} \circ \exp \mathcal{C}_+)\widehat Y$, which yields $\widetilde Y^{\mathrm{up}}(z,u) \sim$

$\widetilde{Y}^{\rm low}\big(z,(\xi_+)^{-1}(u)\big)$ in $\mathcal{D}_+(R,\varepsilon)$. Similarly, $\widetilde{Y}^{\rm low}(z,u)\sim\widetilde{Y}^{\rm up}\big(z,(\xi_-)^{-1}(u)\big)$ in the domain $\mathcal{D}_-(R,\varepsilon)$. When interpreting these relations componentwise with respect to u and modulo $O(|{\rm e}^{\pm(M+\sigma)z}|)$ in $\mathcal{D}_\pm(R,\varepsilon)$ with arbitrarily large M, we get the desired relations between $\widetilde{\varphi}^{\rm up}(z,y)=\widetilde{Y}^{\rm up}(z,y{\rm e}^{-z})$ and $\widetilde{\varphi}^{\rm low}(z,y)=\widetilde{Y}^{\rm low}(z,y{\rm e}^{-z})$.

12 The resurgence monomials $\widetilde{\mathcal{U}}_a^{\omega}$'s and the freeness of alien derivations

12.1. The first goal of this section is to construct families of simple resurgent functions which form closed systems for multiplication and alien derivations in the following sense:

Definition 12.1. We call Δ-friendly monomials the members of any family of simple resurgent functions $(\widetilde{\mathcal{U}}^{\omega_1,\dots,\omega_r})_{r\ge0,\,\omega_1,\dots,\omega_r\in\mathbb{Z}^*}$, such that on the one hand

$$\Delta_m\widetilde{\mathcal{U}}^{\omega_1,\dots,\omega_r}=\left|\begin{array}{ll}\widetilde{\mathcal{U}}^{\omega_2,\dots,\omega_r} & \text{if } r\ge1 \text{ and } \omega_1=m,\\ 0 & \text{if not,}\end{array}\right. \tag{12.1}$$

for every $m\in\mathbb{Z}^*$, and on the other hand $\widetilde{\mathcal{U}}^{\emptyset}=1$ and

$$\widetilde{\mathcal{U}}^{\boldsymbol{\alpha}}\widetilde{\mathcal{U}}^{\boldsymbol{\beta}}=\sum_{\boldsymbol{\omega}\in\Omega^\bullet}\operatorname{sh}\binom{\boldsymbol{\alpha},\ \boldsymbol{\beta}}{\boldsymbol{\omega}}\widetilde{\mathcal{U}}^{\boldsymbol{\omega}},\qquad \boldsymbol{\alpha},\boldsymbol{\beta}\in(\mathbb{Z}^*)^\bullet,$$

i.e., when viewed as a mould, $\widetilde{\mathcal{U}}^\bullet\in\mathcal{M}^\bullet(\mathbb{Z}^*,\widetilde{\mathrm{RES}}_{\mathbb{Z}}^{\rm simp})$ is symmetral.

J. Écalle calls Δ-friendly such resurgent functions by contrast with the functions $\widetilde{\mathcal{V}}_a^{\omega_1,\dots,\omega_r}$, which can be termed "$\partial$-friendly monomials" because of (9.4) (using ∂ as short-hand for $\frac{\rm d}{{\rm d}z}$).

As a matter of fact, Δ-friendly monomials will be defined with the help of the moulds $\widetilde{\mathcal{V}}_a^\bullet$, $\mathcal{V}_a^\bullet$ of Section 9 and mould composition, but we first need to enlarge slightly the definition of mould composition.

12.2. We thus begin with a kind of addendum to Sections 4 and 5. Assume that $\boldsymbol{A}$ is a commutative $\mathbb{C}$-algebra, the unit of which is denoted 1, and Ω is a commutative monoid, the operation of which is denoted additively. We still use the notations $\|\boldsymbol{\omega}\|=\omega_1+\cdots+\omega_r$ and $\|\emptyset\|=0$.

Let us call *restricted moulds* the elements of $\mathcal{M}^\bullet(\Omega^*,\boldsymbol{A})$, where $\Omega^*=\Omega\setminus\{0\}$. The example we have in mind is $\Omega=\mathbb{Z}$ and $\boldsymbol{A}=\mathbb{C}$ or $\widetilde{\mathrm{RES}}_{\mathbb{Z}}^{\rm simp}$.

Definition 12.2. We call *licit mould* any restricted mould $U^\bullet$ such that

$$\|\boldsymbol{\omega}\|=0\implies U^{\boldsymbol{\omega}}=0$$

for any $\boldsymbol{\omega}\in(\Omega^*)^\bullet$. The set of licit moulds will be denoted $\mathcal{M}^\bullet_{\rm lic}(\Omega^*,\boldsymbol{A})$.

The set $\mathcal{M}^\bullet_{\mathrm{lic}}(\Omega^*, \boldsymbol{A})$ is clearly an $\boldsymbol{A}$-submodule of $\mathcal{M}^\bullet(\Omega^*, \boldsymbol{A})$, but not an $\boldsymbol{A}$-subalgebra. Notice that $U^\bullet \in \mathcal{M}^\bullet_{\mathrm{lic}}(\Omega^*, \boldsymbol{A})$ implies $U^\emptyset = 0$.

We now define the *composition of a restricted mould and a licit mould* as follows:

$$(M^\bullet, U^\bullet) \in \mathcal{M}^\bullet(\Omega^*, \boldsymbol{A}) \times \mathcal{M}^\bullet_{\mathrm{lic}}(\Omega^*, \boldsymbol{A}) \mapsto C^\bullet = M^\bullet \circ U^\bullet \in \mathcal{M}^\bullet(\Omega^*, \boldsymbol{A}),$$

with $C^\emptyset = M^\emptyset$ and, for $\boldsymbol{\omega} \neq \emptyset$,

$$C^{\boldsymbol{\omega}} = \sum_{\substack{s \geq 1,\, \boldsymbol{\omega} = \boldsymbol{\omega}^1 \cdots \boldsymbol{\omega}^s \\ \|\boldsymbol{\omega}^1\|, \dots, \|\boldsymbol{\omega}^s\| \neq 0}} M^{(\|\boldsymbol{\omega}^1\|, \dots, \|\boldsymbol{\omega}^s\|)} U^{\boldsymbol{\omega}^1} \cdots U^{\boldsymbol{\omega}^s}.$$

The map $M^\bullet \mapsto M^\bullet \circ U^\bullet$ is clearly $\boldsymbol{A}$-linear; we leave it to the reader[15] to check that it is an $\boldsymbol{A}$-algebra homomorphism, that

$$U^\bullet, V^\bullet \in \mathcal{M}^\bullet_{\mathrm{lic}}(\Omega^*, \boldsymbol{A}) \implies U^\bullet \circ V^\bullet \in \mathcal{M}^\bullet_{\mathrm{lic}}(\Omega^*, \boldsymbol{A}),$$

and that

$$M^\bullet \in \mathcal{M}^\bullet(\Omega^*, \boldsymbol{A}) \text{ and } U^\bullet, V^\bullet \in \mathcal{M}^\bullet_{\mathrm{lic}}(\Omega^*, \boldsymbol{A}) \implies (M^\bullet \circ U^\bullet) \circ V^\bullet = M^\bullet \circ (U^\bullet \circ V^\bullet).$$

The *restricted identity mould* is

$$I^\bullet_* \colon \boldsymbol{\omega} \in (\Omega^*)^\bullet \mapsto I^{\boldsymbol{\omega}}_* = \begin{cases} 1 & \text{if } r(\boldsymbol{\omega}) = 1, \\ 0 & \text{if } r(\boldsymbol{\omega}) \neq 1. \end{cases}$$

It is a licit mould, which satisfies $M^\bullet \circ I^\bullet_* = M^\bullet$ for any restricted mould $M^\bullet$ and $I^\bullet_* \circ U^\bullet = U^\bullet$ for any licit mould $U^\bullet$. One can check that a licit mould $U^\bullet$ admits a licit inverse for composition iff $U^{\boldsymbol{\omega}}$ is invertible in $\boldsymbol{A}$ whenever $r(\boldsymbol{\omega}) = 1$.

A proposition analogous to Proposition 5.3 holds. In particular, *alternal invertible licit moulds form a subgroup of the composition group of invertible licit moulds*[16] and *the composition with an alternal licit mould preserves symmetrality and alternality*. This will be used in the next paragraph.

12.3. We now take $\Omega = \mathbb{Z}$ and $\boldsymbol{A} = \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$. Assume that $a = (\hat{a}_\eta)_{\eta \in \mathbb{Z}^*}$ is any family of entire functions such that $\hat{a}_\eta(\eta) \neq 0$ for each $\eta \in \mathbb{Z}^*$. We still use the notations $\tilde{a}_\eta = \mathcal{B}^{-1}\hat{a}_\eta \in z^{-1}\mathbb{C}[[z^{-1}]]$ and $J^{\boldsymbol{\omega}}_a = \tilde{a}_\eta$ if $\boldsymbol{\omega} = (\eta)$, 0 if not. We recall that, according to Section 9, the equation

$$(\partial + \nabla)\widetilde{\mathcal{V}}^\bullet_a = -\widetilde{\mathcal{V}}^\bullet_a \times J^\bullet_a \tag{12.2}$$

[15] The verification of most of the properties indicated in this paragraph can be simplified by observing that the canonical restriction map $\rho \colon \mathcal{M}^\bullet(\Omega, \boldsymbol{A}) \to \mathcal{M}^\bullet(\Omega^*, \boldsymbol{A})$ is an $\boldsymbol{A}$-algebra homomorphism which satisfies $\rho(M^\bullet \circ U^\bullet) = \rho(M^\bullet) \circ \rho(U^\bullet)$ for any two moulds $M^\bullet$ and $U^\bullet$ such that $\rho(U^\bullet)$ is licit and which preserves alternality and symmetrality.

[16] This can be checked by means of the restriction homomorphism of the previous footnote: if $U^\bullet$ is licit and $U^{\boldsymbol{\omega}}$ is invertible whenever $r(\boldsymbol{\omega}) = 1$, then any $U^\bullet_0 \in \mathcal{M}^\bullet(\Omega, \boldsymbol{A})$ such that $\rho(U^\bullet_0) = U^\bullet$ and $U^{(0)}_0 = 1$ is an invertible mould, the composition inverse of which has a restriction $V^\bullet$ which satisfies $U^\bullet \circ V^\bullet = V^\bullet \circ U^\bullet = I^\bullet_*$; if moreover $U^\bullet$ is alternal, then one can choose $U^\bullet_0$ alternal (take $U^{\boldsymbol{\omega}}_0 = 0$ whenever $r(\boldsymbol{\omega}) \geq 2$ and one of the letters of $\boldsymbol{\omega}$ is 0), thus its inverse and the restriction of its inverse are alternal.

defines a symmetral mould $\widetilde{\mathcal{V}}_a^\bullet \in \mathcal{M}^\bullet(\Omega^*, \boldsymbol{A})$, and that, for each $m \in \mathbb{Z}^*$, we have an alternal scalar mould $V_a^\bullet(m) = -V_a^\bullet(m) \in \mathcal{M}^\bullet(\Omega^*, \mathbb{C})$ which satisfies

$$\Delta_m \widetilde{\mathcal{V}}_a^\bullet = V_a^\bullet(m) \times \widetilde{\mathcal{V}}_a^\bullet, \tag{12.3}$$

$$V_a^{\boldsymbol{\omega}}(m) \neq 0 \implies \|\boldsymbol{\omega}\| = m. \tag{12.4}$$

Moreover $V_a^{(\eta)}(\eta) = 2\pi \mathrm{i}\, \hat{a}_\eta(\eta)$.

Theorem 7. *The formula $V_a^\bullet = \sum_{m\in\mathbb{Z}^*} V_a^\bullet(m)$ defines an alternal scalar licit mould, which admits a composition inverse $U_a^\bullet$. The formula*

$$\widetilde{\mathcal{U}}_a^\bullet = \widetilde{\mathcal{V}}_a^\bullet \circ U_a^\bullet \in \mathcal{M}^\bullet(\Omega^*, \widetilde{\mathrm{RES}}_{\mathbb{Z}}^{\mathrm{simp}}) \tag{12.5}$$

defines a family of Δ-friendly monomials $\widetilde{\mathcal{U}}_a^{\boldsymbol{\omega}}$.

Proof. In view of (12.4), the definition of $V_a^\bullet$ makes sense and its alternality follows from the alternality of each $V_a^\bullet(m)$. This mould is clearly licit, and $V_a^{(\eta)} = 2\pi \mathrm{i}\, \hat{a}_\eta(\eta) \neq 0$, hence its invertibility.

The general properties of the composition of a restricted mould and a licit mould ensure that (12.5) defines a symmetral mould. Its alien derivatives are easily computed since $U_a^\bullet$ is a scalar mould:

$$\Delta_m \widetilde{\mathcal{U}}_a^\bullet = (\Delta_m \widetilde{\mathcal{V}}_a^\bullet) \circ U_a^\bullet = (V_a^\bullet(m) \times \widetilde{\mathcal{V}}_a^\bullet) \circ U_a^\bullet = I_m^\bullet \times \widetilde{\mathcal{U}}_a^\bullet,$$

with $I_m^\bullet = V_a^\bullet(m) \circ U_a^\bullet$ (the last identity follows from the $\boldsymbol{A}$-algebra homomorphism property of post-composition with $U_a^\bullet$). The conclusion follows from the fact that

$$I_m^{\boldsymbol{\omega}} = 1 \ \text{ if } \boldsymbol{\omega} = (m), \quad 0 \ \text{ if not.} \tag{12.6}$$

This formula can be checked by introducing the map $\rho_m \colon M^\bullet \in \mathcal{M}^\bullet(\Omega^*, \boldsymbol{A}) \mapsto M_m^\bullet \in \mathcal{M}^\bullet(\Omega^*, \boldsymbol{A})$ defined by $M_m^{\boldsymbol{\omega}} = M^{\boldsymbol{\omega}}$ if $\|\boldsymbol{\omega}\| = m$, 0 if not, and observing that $\rho_m(M^\bullet \circ U^\bullet) = \rho_m(M^\bullet) \circ U^\bullet$ for any licit mould $U^\bullet$; thus $I_m^\bullet = \rho_m(V_a^\bullet) \circ U_a^\bullet = \rho_m(I_*^\bullet)$. □

Remark 12.1. An analogous computation yields

$$(\partial + \nabla)\widetilde{\mathcal{U}}_a^\bullet = -\widetilde{\mathcal{U}}_a^\bullet \times \widetilde{K}^\bullet$$

with a licit alternal mould $\widetilde{K}^\bullet \in \mathcal{M}^\bullet(\mathbb{Z}^*, \mathbb{C}[[z^{-1}]])$ defined by $\widetilde{K}^{\boldsymbol{\omega}} = U_a^{\boldsymbol{\omega}}\, \widetilde{a}_{\|\boldsymbol{\omega}\|}$ if $\|\boldsymbol{\omega}\| \neq 0$.

12.4. Let $\boldsymbol{\Delta}_\emptyset = \mathrm{Id}$ and $\boldsymbol{\Delta}_{\boldsymbol{m}} = \Delta_{m_r} \cdots \Delta_{m_1}$ for $\boldsymbol{m} = (m_1, \ldots, m_r) \in (\mathbb{Z}^*)^\bullet$ non-empty. As an application of the existence of Δ-friendly monomials, we now show

Theorem 8. *Let $\boldsymbol{A} = \mathrm{RES}_{\mathbb{Z}}^{\mathrm{simp}}$. The family $(\boldsymbol{\Delta}_{\boldsymbol{m}})_{\boldsymbol{m}\in(\mathbb{Z}^*)^\bullet}$ is free in the $\boldsymbol{A}$-module* $\mathrm{End}_{\mathbb{C}}\, \boldsymbol{A}$.

Observe that the free $\boldsymbol{A}$-module generated by the $\boldsymbol{\Delta}_{\boldsymbol{m}}$'s is a subring of $\mathrm{End}_{\mathbb{C}}\, \boldsymbol{A}$ (because the Δ_m's are derivations).

Proof. We must show that there is no non-trivial polynomial relation between the alien derivations Δ_m. Consider a non-commutative polynomial with coefficients in $\boldsymbol{A}$,

$$P = \sum_{\boldsymbol{\omega}\in\mathcal{F}} \widetilde{\varphi}^{\boldsymbol{\omega}} \boldsymbol{\Delta}_{\boldsymbol{\omega}}, \quad \mathcal{F} \text{ finite subset of } (\mathbb{Z}^*)^{\bullet},$$

and assume that not all the coefficients are zero; we shall prove that $P \neq 0$.

We may suppose $\mathcal{F} \neq \emptyset$ and $\widetilde{\varphi}^{\boldsymbol{\omega}} \neq 0$ for each $\boldsymbol{\omega} \in \mathcal{F}$. Choose $\boldsymbol{m} = (m_1, \dots, m_r) \in \mathcal{F}$ with minimal length; then, for any family of Δ-friendly monomials $\widetilde{\mathcal{U}}^{\bullet}$, we find $P\widetilde{\mathcal{U}}^{\boldsymbol{m}} = \widetilde{\varphi}^{m_1,\dots,m_r} \neq 0$ as a consequence of

$$\Delta_{m_s} \cdots \Delta_{m_1} \widetilde{\mathcal{U}}^{\boldsymbol{\omega}} = \begin{cases} \widetilde{\mathcal{U}}^{\boldsymbol{n}} & \text{if } \boldsymbol{\omega} = (m_1, \dots, m_s)\raisebox{0.2ex}{\scriptsize$\bullet$}\boldsymbol{n} \text{ with } \boldsymbol{n} \in (\mathbb{Z}^*)^{\bullet}, \\ 0 & \text{if not.} \end{cases} \tag{12.7}$$

□

12.5. Let us call *resurgence constant* any $\widetilde{\varphi} \in \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ such that $\Delta_m \widetilde{\varphi} = 0$ for any $m \in \mathbb{Z}^*$. This is equivalent to saying that $\mathcal{B}\widetilde{\varphi} = c\,\delta + \widehat{\varphi}(\zeta)$ with $c \in \mathbb{C}$ and $\widehat{\varphi}$ entire (in particular every convergent series $\widetilde{\varphi}(z) \in \mathbb{C}\{z^{-1}\}$ is a resurgence constant, but the converse is not true since we did not require the Borel transform to be of exponential type: the entire function $\widehat{\varphi}$ might have order > 1).

Resurgence constants form a subalgebra $\widetilde{\mathcal{P}}_0$ of $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$.

Proposition 12.1. *Let $\widetilde{\mathcal{U}}^{\bullet}_1$ and $\widetilde{\mathcal{U}}^{\bullet}_2$ be two moulds in $\mathcal{M}^{\bullet}(\Omega^*, \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}})$ and suppose that $\widetilde{\mathcal{U}}^{\bullet}_1$ is a family of Δ-friendly monomials. Then $\widetilde{\mathcal{U}}^{\bullet}_2$ is a family of Δ-friendly monomials iff if there exists a symmetral mould $\widetilde{M}^{\bullet} \in \mathcal{M}^{\bullet}(\mathbb{Z}^*, \widetilde{\mathcal{P}}_0)$ such that*

$$\widetilde{\mathcal{U}}^{\bullet}_2 = \widetilde{\mathcal{U}}^{\bullet}_1 \times \widetilde{M}^{\bullet}. \tag{12.8}$$

Thus all the families of Δ-friendly monomials can be deduced from one of them.

Proof. Let $\widetilde{M}^{\bullet} = (\widetilde{\mathcal{U}}^{\bullet}_1)^{-1} \times \widetilde{\mathcal{U}}^{\bullet}_2 \in \mathcal{M}^{\bullet}(\Omega^*, \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}})$. This mould is symmetral iff $\widetilde{\mathcal{U}}^{\bullet}_2$ is symmetral. Let $m \in \mathbb{Z}^*$. We have $\Delta_m \widetilde{\mathcal{U}}^{\bullet}_1 = I^{\bullet}_m \times \widetilde{\mathcal{U}}^{\bullet}_1$, with the mould $I^{\bullet}_m$ defined by (12.6). The Leibniz rule applied to (12.8) yields $\Delta_m \widetilde{\mathcal{U}}^{\bullet}_2 = I^{\bullet}_m \times \widetilde{\mathcal{U}}^{\bullet}_2 + \widetilde{\mathcal{U}}^{\bullet}_1 \times \Delta_m \widetilde{M}^{\bullet}$. Thus $\widetilde{\mathcal{U}}^{\bullet}_2$ satisfies (12.1) for all m iff $\widetilde{\mathcal{U}}^{\bullet}_1 \times \Delta_m \widetilde{M}^{\bullet} = 0$ for all m, which is equivalent to $\widetilde{M}^{\boldsymbol{\omega}} \in \widetilde{\mathcal{P}}_0$ since $\widetilde{\mathcal{U}}^{\bullet}_1$ admits a multiplicative inverse. □

12.6. We call *resurgence polynomial* any $\widetilde{\varphi} \in \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ such that $\boldsymbol{\Delta}_{\boldsymbol{\omega}} \widetilde{\varphi} = 0$ for all but finitely many $\boldsymbol{\omega} \in (\mathbb{Z}^*)^{\bullet}$. Resurgence polynomials form a subalgebra $\widetilde{\mathcal{P}}$ of $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ (which contains $\widetilde{\mathcal{P}}_0$).

Proposition 12.2. *Let $\widetilde{\mathcal{U}}^{\bullet}$ be any family of Δ-friendly monomials and $\widetilde{\varphi}$ be any simple resurgent function. Then $\widetilde{\varphi}$ is a resurgence polynomial iff $\widetilde{\varphi}$ can be written as*

$$\widetilde{\varphi} = \sum_{\boldsymbol{\omega}\in\mathcal{F}} \widetilde{\mathcal{U}}^{\boldsymbol{\omega}} \widetilde{\varphi}_{\boldsymbol{\omega}}, \quad \mathcal{F} \textit{ finite subset of } (\mathbb{Z}^*)^{\bullet}, \tag{12.9}$$

with $\widetilde{\varphi}_{\boldsymbol{\omega}} \in \widetilde{\mathcal{P}}_0$ for every $\boldsymbol{\omega} \in \mathcal{F}$. Moreover, such a representation of a resurgence polynomial is unique and the formula $\mathcal{E} = \sum S\widetilde{\mathcal{U}}^\bullet \boldsymbol{\Delta}_\bullet$ (with S defined by (5.3), thus $S\widetilde{\mathcal{U}}^\bullet$ is the multiplicative inverse of $\widetilde{\mathcal{U}}^\bullet$) defines an algebra homomorphism $\mathcal{E}\colon \widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}}_0$ such that

$$\widetilde{\varphi}_{\boldsymbol{\omega}} = \mathcal{E}\boldsymbol{\Delta}_{\boldsymbol{\omega}}\widetilde{\varphi}, \quad \boldsymbol{\omega} \in (\mathbb{Z}^*)^\bullet.$$

Proof. In view of (12.7), formula (12.9) defines a resurgence polynomial whenever the $\widetilde{\varphi}_{\boldsymbol{\omega}}$'s are resurgence constants.

The formula $\mathcal{E} = \sum S\widetilde{\mathcal{U}}^\bullet \boldsymbol{\Delta}_\bullet$ makes sense as an operator $\widetilde{\mathcal{P}} \to \widetilde{\mathrm{RES}}^{\mathrm{simp}}_{\mathbb{Z}}$ since the sum is locally finite; an easy adaptation of the arguments of Section 7 shows that $\mathcal{E}$ is an algebra homomorphism because $\widetilde{\mathcal{U}}^\bullet$ is symmetral and $\boldsymbol{\Delta}_\bullet$ can be viewed as a cosymmetral comould (the Δ_m's which generate it are derivations of $\widetilde{\mathcal{P}}$).

Let us check that $\mathcal{E}(\widetilde{\mathcal{P}}) \subset \widetilde{\mathcal{P}}_0$. Let $\widetilde{\varphi} \in \widetilde{\mathcal{P}}$ and $m \in \mathbb{Z}^*$; we can write $\Delta_m = \sum I_m^\bullet \boldsymbol{\Delta}_\bullet$ with the notation (12.6). A computation analogous to the proof of Proposition 6.1, but taking into account the fact that Δ_m does not commute with the multiplication by $(S\widetilde{\mathcal{U}})^{\boldsymbol{\omega}}$, shows that

$$\Delta_m \mathcal{E}\widetilde{\varphi} = \sum \big((S\widetilde{\mathcal{U}} \times I_m^\bullet) + \Delta_m S\widetilde{\mathcal{U}}^\bullet\big)\boldsymbol{\Delta}_\bullet\widetilde{\varphi}.$$

Since $\Delta_m\widetilde{\mathcal{U}}^\bullet = I_m^\bullet \times \widetilde{\mathcal{U}}^\bullet$ and S is an anti-homomorphism such that $SI_m^\bullet = -I_m^\bullet$ and $S\Delta_m = \Delta_m S$, we have $\Delta_m S\widetilde{\mathcal{U}}^\bullet = -S\widetilde{\mathcal{U}} \times I_m^\bullet$, hence $\Delta_m\mathcal{E}\widetilde{\varphi} = 0$.

We conclude by considering $\widetilde{\varphi} \in \widetilde{\mathcal{P}}$ and setting $\widetilde{\varphi}_{\boldsymbol{\alpha}} = \mathcal{E}\boldsymbol{\Delta}_{\boldsymbol{\alpha}}\widetilde{\varphi}$ for every word $\boldsymbol{\alpha} \in (\mathbb{Z}^*)^\bullet$ (but only finitely many words may yield a nonzero result). We have $\widetilde{\varphi}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}} (S\widetilde{\mathcal{U}}^\bullet)^{\boldsymbol{\beta}} \boldsymbol{\Delta}_{\boldsymbol{\alpha}\boldsymbol{\cdot}\boldsymbol{\beta}}\widetilde{\varphi}$, thus $\sum_{\boldsymbol{\alpha}} \widetilde{\mathcal{U}}^{\boldsymbol{\alpha}}\widetilde{\varphi}_{\boldsymbol{\alpha}} = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \widetilde{\mathcal{U}}^{\boldsymbol{\alpha}}(S\widetilde{\mathcal{U}}^\bullet)^{\boldsymbol{\beta}}\boldsymbol{\Delta}_{\boldsymbol{\alpha}\boldsymbol{\cdot}\boldsymbol{\beta}}\widetilde{\varphi}$, and the identity $\widetilde{\mathcal{U}}^\bullet \times S\widetilde{\mathcal{U}}^\bullet = 1^\bullet$ implies $\sum_{\boldsymbol{\alpha}} \widetilde{\mathcal{U}}^{\boldsymbol{\alpha}}\widetilde{\varphi}_{\boldsymbol{\alpha}} = \widetilde{\varphi}$. $\square$

13 Other applications of mould calculus

13.1. In this last section, we wish to indicate how mould calculus can be applied to another classical normal form problem: the linearisation of a vector field with non-resonant spectrum.

Let $\mathcal{A} = \mathbb{C}[[y_1, \dots, y_n]]$ with $n \in \mathbb{N}^*$, and consider a vector field with diagonal linear part:

$$X = \sum_{i=1}^{n} a_i(y)\frac{\partial}{\partial y_i}, \quad a_i(y) = \lambda_i y_i + \sum_{k \in \mathbb{N}^n,\, |k| \geq 2} a_{i,k} y^k$$

(with standard notations for the multi-indices: $y^k = y_1^{k_1} \cdots y_n^{k_n}$ and $|k| = k_1 + \cdots + k_n$ if $k = (k_1, \dots, k_n)$).

The first problem consists in finding a formal transformation which conjugates X and its linear part

$$X^{\mathrm{lin}} = \sum_{i=1}^{n} \lambda_i y_i \frac{\partial}{\partial y_i}.$$

This linear part is thus considered as a natural candidate to be a normal form; it is determined by the spectrum $\lambda = (\lambda_1, \dots, \lambda_n)$. In fact $X^{\mathrm{lin}} = \mathcal{X}_\lambda$ with the notation (6.4).

It is not always possible to find a formal conjugacy between X and X^{lin}, because elementary calculations let appear rational functions of the spectrum, the denominators of which are of the form

$$\langle m, \lambda \rangle = m_1\lambda_1 + \cdots + m_n\lambda_n \tag{13.1}$$

with certain multi-indices $m \in \mathbb{Z}^n$. Let us make the following *strong non-resonance assumption*:

$$\langle m, \lambda \rangle \neq 0 \quad \text{for every } m \in \mathbb{Z}^n \setminus \{0\}. \tag{13.2}$$

We shall now indicate how to construct a formal conjugacy via mould-comould expansions under this assumption.

13.2. We are in the framework of Section 6 with $\boldsymbol{A} = \mathbb{C}$. Let us use the standard monomial valuation on $\mathcal{A}$, defined by $\nu(y^k) = |k|$. We shall manipulate operators of $\mathcal{A}$ having a valuation with respect to ν; they form a subspace $\mathcal{F}$ of $\operatorname{End}_{\mathbb{C}} \mathcal{A}$ which was denoted $\mathcal{F}_{\mathcal{A},\boldsymbol{A}}$ in (6.2).

We first decompose X as a sum of homogeneous components, in the sense of Definition 6.2: X^{lin} is homogeneous of degree 0 and we can write

$$X - X^{\mathrm{lin}} = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} a_{i,k}\, y^k \frac{\partial}{\partial y_i},$$

thus extending the definition of the $a_{i,k}$'s:

$$a_{i,k} \neq 0 \implies k \in \mathbb{N}^n \text{ and } |k| \geq 2.$$

Using the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{Z}^n$, we can write

$$X - X^{\mathrm{lin}} = \sum_{m \in \mathbb{Z}^n} B_m, \quad B_m = \sum_{i=1}^{n} a_{i,m+e_i}\, y^m \cdot y_i \frac{\partial}{\partial y_i}. \tag{13.3}$$

Observe that each B_m is homogeneous of degree $m \in \mathbb{Z}^n$ and that

$$B_m \neq 0 \implies m \in \mathcal{N},$$
$$\mathcal{N} = \{\, m \in \mathbb{Z}^n \mid m + e_i \in \mathbb{N}^m \text{ for some } i \text{ and } |m| \geq 1 \,\}.$$

We thus view $\mathcal{N}$ as an alphabet and consider

$$\boldsymbol{B}_\emptyset = \mathrm{Id}, \quad \boldsymbol{B}_{m_1,\dots,m_r} = B_{m_r} \cdots B_{m_1}$$

as a comould on $\mathcal{N}$ with values in $\mathcal{F}$. For instance, $X - X^{\text{lin}} = \sum I^\bullet \boldsymbol{B}_\bullet$. The inequalities

$$\operatorname{val}_\nu\left(\boldsymbol{B}_{m_1,\dots,m_r}\right) \geq |m_1 + \cdots + m_r|$$

show that, for any scalar mould $M^\bullet \in \mathcal{M}^\bullet(\mathcal{N}, \mathbb{C})$, the family $(M^{\boldsymbol{m}} \boldsymbol{B}_{\boldsymbol{m}})_{\boldsymbol{m} \in \mathcal{N}^\bullet}$ is formally summable in $\mathcal{F}$ (indeed, for any $\delta \in \mathbb{Z}$, $\operatorname{val}_\nu\left(M^{m_1,\dots,m_r} \boldsymbol{B}_{m_1,\dots,m_r}\right) \leq \delta$ implies $r \leq |m_1| + \cdots + |m_r| \leq \delta$ and there are only finitely many $\eta \in \mathcal{N}$ such that $|\eta| \leq \delta$).

13.3. According to the general strategy of mould-comould expansions, we now look for a formal conjugacy θ between X and X^{lin} through its substitution automorphism Θ, which should satisfy $X = \Theta^{-1} X^{\text{lin}} \Theta$. This conjugacy equation can be rewritten

$$\left[X^{\text{lin}}, \Theta\right] = \Theta\left(X - X^{\text{lin}}\right).$$

Propositions 6.1 and 6.3 show that, given any $M^\bullet \in \mathcal{M}^\bullet(\mathcal{N}, \mathbb{C})$, $\Theta = \sum M^\bullet \boldsymbol{B}_\bullet$ is solution as soon as

$$D_\varphi M^\bullet = I^\bullet \times M^\bullet, \tag{13.4}$$

with $D_\varphi M^{\boldsymbol{m}} = \langle \|\boldsymbol{m}\|, \lambda\rangle M^{\boldsymbol{m}}$ for $\boldsymbol{m} \in \mathcal{N}^\bullet$.

Assumption (13.2) allows us to find a unique solution of equation (13.4) such that $M^\emptyset = 1$; it is inductively determined by

$$M^{m_1,\dots,m_r} = \frac{1}{\langle \|\boldsymbol{m}\|, \lambda\rangle} M^{m_2,\dots,m_r},$$

hence

$$M^{\boldsymbol{m}} = \frac{1}{\langle m_1 + \cdots + m_r, \lambda\rangle} \frac{1}{\langle m_2 + \cdots + m_r, \lambda\rangle} \cdots \frac{1}{\langle m_r, \lambda\rangle} \tag{13.5}$$

The symmetrality of this solution can be obtained by mimicking the proof of Proposition 5.5.

Since $\boldsymbol{B}_\bullet$ is cosymmetral, we thus have an automorphism $\Theta = \sum M^\bullet \boldsymbol{B}_\bullet$; since Θ is continuous for the Krull topology, $\theta = (\theta_1, \dots, \theta_n)$ with $\theta_i = \Theta y_i$ yields a formal tangent-to-identity transformation which conjugates X and X^{lin}.

13.4. As was alluded to at the end of Section 7, the formalism of moulds can be equally applied to the normalisation of discrete dynamical systems. A problem parallel to the previous one is the linearisation of a formal transformation with multiplicatively non-resonant spectrum.

Suppose indeed that $f = (f_1, \dots, f_n)$ is a n-tuple of formal series of $\mathcal{A}$ without constant terms, with diagonal linear part $f^{\text{lin}}\colon (y_1, \dots, y_n) \mapsto (\ell_1 y_1, \dots, \ell_n y_n)$. Conjugating f and f^{lin} is equivalent to finding a continuous automorphism Θ which conjugates the corresponding substitution automorphisms: $F = \Theta^{-1} F^{\text{lin}} \Theta$.

This is possible under the following *strong multiplicative non-resonance assumption* on the spectrum $\ell = (\ell_1, \dots, \ell_n)$:

$$\ell^m - 1 \neq 0 \quad \text{for every } m \in \mathbb{Z}^n \setminus \{0\}. \tag{13.6}$$

An explicit solution is obtained by expanding $F(F^{\mathrm{lin}})^{-1}$ in homogeneous components

$$F = \Big(\mathrm{Id} + \sum_{m\in\mathcal{N}} B_m\Big) F^{\mathrm{lin}},$$

where the homogeneous operators B_m are no longer derivations; instead, they satisfy the modified Leibniz rule (7.4) and generate a *cosymmetrel* comould $\boldsymbol{B}_\bullet$. Correspondingly, the scalar mould

$$M^{\emptyset} = 1, \quad M^{\boldsymbol{m}} = \frac{1}{(\ell^{m_1+\cdots+m_r}-1)(\ell^{m_2+\cdots+m_r}-1)\cdots(\ell^{m_r}-1)} \tag{13.7}$$

is *symmetrel* and $\Theta = \sum M^\bullet \boldsymbol{B}_\bullet$ is the desired automorphism (see [7]), whence a formal tangent-to-identity transformation θ which conjugates f and f^{lin}.

13.5. In both previous problems, it is a classical result that a formal linearising transformation θ exists under a weaker non-resonance assumption: namely, it is sufficient that (13.2) or (13.6) hold with $\mathbb{Z}^n \setminus \{0\}$ replaced by $\mathcal{N} \setminus \{0\}$. Unfortunately, this is not clear on the mould-comould expansion, since under this weaker assumption the formula (13.5) or (13.7) may involve a zero divisor, thus the mould $M^\bullet$ is not well-defined.

J. Écalle has invented a technique called *arborification* which solves this problem and which goes far beyond: arborification also allows to recover the Bruno–Rüssmann theorem, according to which the formal linearisation θ is convergent whenever the vector field X or the transformation f is convergent and the spectrum λ or ℓ satisfies the so-called *Bruno condition* (a Diophantine condition which states that the divisors $\langle m, \lambda\rangle$ or $\ell^m - 1$ do not approach zero "abnormally well").

The point is that, even when X or f is convergent and the spectrum is Diophantine, it is hard to check that $\theta_i(y)$ is convergent because it is represented as the sum of a formally summable family $(M^{\boldsymbol{m}}\boldsymbol{B}_{\boldsymbol{m}} y_i)_{\boldsymbol{m}\in\mathcal{N}^\bullet}$ in $\mathbb{C}[[y_1,\dots,y_n]]$, but the family $\big(|M^{\boldsymbol{m}}\boldsymbol{B}_{\boldsymbol{m}} y_i|\big)_{\boldsymbol{m}\in\mathcal{N}^\bullet}$ may fail to be summable in $\mathbb{C}$ for any $y \in \mathbb{C}^n \setminus \{0\}$. However, arborification provides a systematic way of reorganizing the terms of the sum: $\theta_i(y)$ then appears as the sum of a summable family indexed by "arborescent sequences" rather than words. The reader is referred to [5], [6], [11], and also to the recent article [10].

13.6. There is another context, seemingly very different, in which J. Écalle has used mould calculus with great efficiency. The multizeta values

$$\zeta(s_1, s_2, \dots, s_r) = \sum_{n_1>n_2>\cdots>n_r>0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_r^{s_r}}$$

naturally present themselves as a scalar mould on $\mathbb{N}^*$; in fact,

$$\zeta(s_1,\dots,s_r) = \mathrm{Ze}^{\binom{0,\ \dots,\ 0}{s_1,\ \dots,\ s_r}},$$

with

$$\mathrm{Ze}^{\binom{\varepsilon_1,\ \dots,\ \varepsilon_r}{s_1,\ \dots,\ s_r}} = \sum_{n_1 > \dots > n_r > 0} \frac{\mathrm{e}^{2\pi\mathrm{i}(n_1\varepsilon_1 + \dots + n_r\varepsilon_r)}}{n_1^{s_1} \cdots n_r^{s_r}}$$

for $s_1, \dots, s_r \in \mathbb{N}^*$, $\varepsilon_1, \dots, \varepsilon_r \in \mathbb{Q}/\mathbb{Z}$ (with a suitable convention to handle possible divergences). The mould $\mathrm{Ze}^\bullet$ is the central object; it turns out that it is *symmetrel*. It is called a *bimould* because the letters of the alphabet are naturally given as members of a product space, here $\mathbb{N}^* \times (\mathbb{Q}/\mathbb{Z})$; this makes it possible to define new operations and structures. This is the starting point of a whole theory, aimed at describing the algebraic structures underlying the relations between multizeta values.

There is a related mould on $\{0\} \cup \exp(2\pi\mathrm{i}\mathbb{Q})$, defined by

$$\mathrm{Wa}^{\alpha_1,\dots,\alpha_\ell} = (-1)^{\ell_0} \int \cdots \int_{0<\zeta_1<\dots<\zeta_\ell<1} \frac{\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_\ell}{(\alpha_1 - \zeta_1) \cdots (\alpha_\ell - \zeta_\ell)},$$

where the α_j's are either 0 or roots of unity and ℓ_0 is the number of 0's among them. With a suitable extension of this definition when $\alpha_1 = 0$ or $\alpha_\ell = 1$ (in which case the above integral is divergent), one gets a *symmetral* mould, which is related to the multiple logarithms $L^\bullet$ defined by (9.19) (themselves closely related to the moulds $\mathcal{V}_a^\bullet$ and $V_a^\bullet$ with $\hat{a}_\eta \equiv 1$ for every η):

$$\mathrm{Wa}^{\alpha_1,\dots,\alpha_\ell} = \frac{1}{2\pi\mathrm{i}} (-1)^{\ell-\ell_0} L^{\alpha_1,\alpha_2-\alpha_1,\dots,\alpha_\ell-\alpha_{\ell-1},1-\alpha_\ell}$$

(at least when $\alpha_j \in \{-1, 0, 1\}$, if not the definition of $L^\bullet$ must be extended).

The relation between $\mathrm{Ze}^\bullet$ and $\mathrm{Wa}^\bullet$ is

$$\mathrm{Ze}^{\binom{\varepsilon_1,\ \dots,\ \varepsilon_r}{s_1,\ \dots,\ s_r}} = \mathrm{Wa}^{\widehat{e}_r, 0^{[s_r-1]}, \dots, \widehat{e}_1, 0^{[s_1-1]}},$$

with $\widehat{e}_j = \mathrm{e}^{2\pi\mathrm{i}(\varepsilon_1 + \dots + \varepsilon_j)}$.

The reader is referred to [8] or [9].

Acknowledgements. It is a pleasure to thank F. Fauvet, F. Menous, F. Patras and B. Teissier for their help. I am also indebted to F. Fauvet and J.-P. Ramis for having organized the conference “Renormalization and Galois theories” and given me a great opportunity of lecturing on resurgence theory, and to J.-P. Rolin for his scrutinising reading of the manuscript. This work was done in the framework of the project *Phénomène de Stokes, renormalisation, théories de Galois* from the *agence nationale pour la recherche*.

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Galois theory, motives and transcendental numbers

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Abstract. From its early beginnings up to nowadays, algebraic number theory has evolved in symbiosis with Galois theory: indeed, one could hold that it consists in the very study of the absolute Galois group of the field of rational numbers.

Nothing like that can be said of transcendental number theory. Nevertheless, could one not associate conjugates and a Galois group to transcendental numbers such as π? Beyond, can one not envision an appropriate Galois theory in the field of transcendental number theory? In which role?

The aim of this text is to indicate what Grothendieck's theory of motives has to say, at least conjecturally, on these questions.

1 The basic question

Let α be an *algebraic* complex number. This means that α is a root of a non-zero polynomial p with rational coefficients. One may assume that p is of minimal degree, say n; this ensures that p has no multiple roots. Its complex roots are called the *conjugates* of α.

The polynomial expressions with rational coefficients in the conjugates of α form a field (the splitting field of p), also called the *Galois closure* of $\mathbb{Q}[\alpha]$. We denote it by $\mathbb{Q}[\alpha]_{\mathrm{gal}}$ and view it as a subring of $\mathbb{C}$.

The *Galois group* of α (or p) is the group of automorphism of the ring $\mathbb{Q}[\alpha]_{\mathrm{gal}}$. We denote it by G_α.

Two fundamental facts of Galois theory are:

(1) G_α identifies with a subgroup of the permutation group of the conjugates of α, and permutes transitively these conjugates,

(2) the elements in $\mathbb{Q}[\alpha]_{\mathrm{gal}}$ fixed by G_α are in $\mathbb{Q}$.

In this paper, we address the following

Basic question. *Is there anything analogous for (some) transcendental numbers?*

*This work was partially supported by ANR Galois.

2 A naive approach

2.1 The case of π

Let us try the following naive idea: π being transcendental, one can expect its "conjugates" to be in infinite number; this suggests to look for a formal power series with rational coefficients as a substitute for the minimal polynomial. There is an obvious choice at hand:

$$\prod_{n\in\mathbb{Z}\setminus 0}\left(1-\frac{x}{n\pi}\right)=\frac{\sin x}{x}\in\mathbb{Q}[[x]],$$

which suggests in turn that the non-zero integral multiple of π are conjugate to π. On the other hand, if one insists to have a Galois group which permutes transitively the conjugates, one is forced to include all non-zero rational multiple of π as well. Whence a tentative answer:

$$\text{set of conjugates of } \pi\colon\ \mathbb{Q}^{\times}.\pi,$$
$$\text{Galois closure: } \mathbb{Q}[\pi]_{\mathrm{gal}}=\mathbb{Q}[\pi].$$
$$\text{Galois group of } \pi\colon\ G_\pi=\mathbb{Q}^{\times},$$

Note that G_π acts transitively on $\mathbb{Q}^{\times}.\pi$ and $\mathbb{Q}[\pi]_{\mathrm{gal}}^{G_\pi}=\mathbb{Q}$.

2.2 The case of elliptic periods

Let us consider a period α attached to an elliptic curve E defined over $\mathbb{Q}$ (it is an old theorem of Schneider that α is transcendental). To fix ideas, let E be given in affine form by the Weierstrass equation

$$y^2=4x^3-g_2x-g_3,\quad g_2,g_3\in\mathbb{Q},$$

and let

$$L=\left\langle\int\frac{dx}{y}\right\rangle=\mathbb{Z}\omega_1\oplus\mathbb{Z}\omega_2$$

be the period lattice. Then $\alpha\in L$, $E(\mathbb{C})\cong\mathbb{C}/L$ and

$$g_2=60\sum_{\omega\in L\setminus 0}\omega^{-4},\quad g_3=140\sum_{\omega\in L\setminus 0}\omega^{-6}.$$

Following the same path as for π, let us consider the product $\prod_{\omega\in L\setminus 0}(1-\frac{x}{\omega})$, or rather its convergent version, which is precisely the Weierstrass sigma function divided by x:

$$\prod_{\omega\in L\setminus 0}\left(1-\frac{x}{\omega}\right)e^{x/\omega+x^2/2\omega^2}=\frac{\sigma(x)}{x}\in\mathbb{Q}[[x]].$$

This suggests that elements $\omega\in L\setminus 0$ are conjugate to α. Again, if one insists to have a Galois group which permutes transitively the conjugates, one is forced to include all non-zero elements of $L_{\mathbb{Q}} := \mathbb{Q}\omega_1 \oplus \mathbb{Q}\omega_2$. Whence a tentative answer:

$$\text{set of conjugates of } \alpha\colon \ L_{\mathbb{Q}} \setminus 0,$$
$$\text{Galois closure: } \mathbb{Q}[\alpha]_{\text{gal}} = \mathbb{Q}[\omega_1, \omega_2].$$

Let us turn to the Galois group G_α. It should be a group of automorphism of the algebra $\mathbb{Q}[\alpha]_{\text{gal}}$ and permute transitively the elements of $L_{\mathbb{Q}} \setminus 0$. Here, one has to distinguish two cases:

(1) *The general case*: End $E_{\mathbb{C}} = \mathbb{Z}$. In this case, it is conjectured that ω_1 and ω_2 are algebraically independent, so that $\mathbb{Q}[\alpha]_{\text{gal}}$ is a polynomial algebra in two variables. For G_α to act transitively on $L_{\mathbb{Q}} \setminus 0$, one must have

$$\text{Galois group of } \alpha\colon \ G_\alpha = \operatorname{Aut} L_{\mathbb{Q}} \cong \mathrm{GL}_2(\mathbb{Q}).$$

Note that, conversely, for Aut $L_{\mathbb{Q}} \cong \mathrm{GL}_2(\mathbb{Q})$ to act on $\mathbb{Q}[\alpha]_{\text{gal}}$, the latter must be a polynomial algebra in two variables.

(2) *The CM case*: End $E_{\mathbb{C}}$ is an order in an imaginary quadratic field K. In this case, $\omega_2/\omega_1 \in K$, so that $K^\times$ acts naturally on $\mathbb{Q}[\alpha]_{\text{gal}}$. In fact, transcendental number theory shows that the algebraicity of ω_2/ω_1 is the only relation in $\mathbb{Q}[\alpha]_{\text{gal}}$, and one derives that Spec $\mathbb{Q}[\alpha]_{\text{gal}}$ is a torsor under the normalizer N_K in Aut $L_{\mathbb{Q}}$ of a Cartan subgroup isomorphic to $K^\times$ (viewed as a 2-dimensional torus over $\mathbb{Q}$). Thus in the CM case, one is led to set

$$\text{Galois group of } \alpha\colon \ G_\alpha = N_K.$$

Note that in both cases G_α acts transitively on $L_{\mathbb{Q}} \setminus 0$ and $\mathbb{Q}[\alpha]_{\text{gal}}^{G_\alpha} = \mathbb{Q}$.

2.3 Generalization?

The following elementary result, due to Hurwitz[1], seems encouraging at first:

For any $\alpha \in \mathbb{C}$, there exists $p \in \mathbb{Q}[[x]] \setminus 0$ which defines an entire function of exponential growth, and vanishes at α.

However, it turns out that there are uncountably many such series p! In fact, such a series can be found which vanishes not only at α, but also at any other fixed number β, so that there is no hope to define conjugates in this way in general! Therefore, this naive approach leads to a dead-end.

Nevertheless, we shall argue in the sequel that the tentative answers found for π and elliptic periods are the right ones, albeit for different reasons. More generally, the aim of this text is to promote the idea, introduced in [3, 23.5], that *periods should have well-defined conjugates and a Galois group which permutes them transitively.*

[1] as I learned from R. Perez-Marco.

3 Periods and motives

3.1 Periods

In this paper, we use the term "periods" in the same sense as in [8]. Namely, an *effective period* is a complex number whose real and imaginary part are absolutely convergent multiple integrals

$$\int_{\Sigma} \Omega$$

where Σ is a domain in $\mathbb{R}^n$ defined by polynomial equations and inequations with rational coefficients, and Ω is a rational differential form with rational coefficients.

One can show that effective periods are nothing but (convergent) integrals of (top degree) differential forms ω on smooth algebraic varieties X defined over $\mathbb{Q}$ (or $\bar{\mathbb{Q}}$, this amounts to the same), integrated over relative chains $\sigma \in H_{\dim X}(X, D)$ (D being a divisor in X, which may be chosen with normal crossings).[2]

It is clear that effective periods form a countable sub-$\mathbb{Q}$-algebra of $\mathbb{C}$ which contains π. One obtains the algebra of *periods* from it by inverting $2\pi i$.

We shall see a number of examples of periods in the sequel. We refer to [8] for many more concrete examples. For instance, the values at algebraic numbers of generalized hypergeometric series ${}_pF_{p-1}$ with rational parameters are periods.

Periods also frequently appear in connection with Feynman integrals: work by Belkale and Brosnan [5], Bogner and Weinzierl [6] shows that Feynman amplitudes $I(\mathrm{D})$ with rational parameters can be written as a product of a Gamma-factor and a meromorphic function $H(\mathrm{D})$ such that the coefficients of its Taylor expansion at any even integer D are periods.

On the other hand, using complexity theory, Yoshinaga has given examples of complex numbers which are not (effective) periods.[3]

3.2 Betti and De Rham cohomologies

If X^∞ is a smooth manifold, rational combinations of cycles give rise, by duality, to Betti (= singular) cohomology $H_{\mathrm{B}}(X^\infty)$ with rational coefficients, whereas smooth complex differential forms give rise to De Rham cohomology $H_{\mathrm{DR}}(X^\infty)$. By De Rham's theorem, integration of forms along cycles then gives rise to an isomorphism

$$H_{\mathrm{DR}}(X^\infty) \cong H_{\mathrm{B}}(X^\infty) \otimes_{\mathbb{Q}} \mathbb{C}.$$

This extends to the relative case (i.e. to relative cohomology $H(X, D)$).

When X is a smooth algebraic variety over a subfield k of C, there is a more algebraic version of this isomorphism, using the notion of algebraic De Rham coho-

[2] An important point, implicit in [8] and proven in [5], is that it is equivalent to consider convergent integrals of differential forms with poles along D, or integrals of differential forms without poles. See also [3, chap. 7, chap. 23] or [4, §1] for more precisions on periods.

[3] This answers a question in [8].

mology $H_{\mathrm{DR}}(X)$: if X is affine, this is just the cohomology of the De Rham complex of algebraic differential forms on X (defined over k). This is a finite-dimensional k-vector space, and a deep theorem of Grothendieck says that integration gives rise to an isomorphism

$$\varpi_X : H_{\mathrm{DR}}(X) \otimes_k \mathbb{C} \cong H_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

A similar isomorphism $\varpi_{X,D}$ exists in relative cohomology. In the special case $k = \mathbb{Q}$, we see that periods are nothing but entries of the matrix of $\varpi_{X,D}$ with respect to some basis of the $\mathbb{Q}$-vector space $H_{\mathrm{DR}}(X)$ (*resp.* $H_{\mathrm{B}}(X)$). This is why ϖ_X or $\varpi_{X,D}$ is often called *the period isomorphism*.

3.3 Motives

A conceptual framework for the study of periods is provided by the theory of motives. There exist several, more or less conjectural[4], versions of this framework, and the choice will not matter here. For more detail, we refer to [3].

Motives are intermediate between algebraic varieties and their linear invariants (cohomology): they are of algebro-geometric nature on one hand, but they are supposed to play the role of a universal cohomology for algebraic varieties and thus to enjoy the same formalism on the other hand.

Here, we restrict our attention to algebraic varieties defined over $\mathbb{Q}$. Let us denote by $\mathrm{Var}(\mathbb{Q})$ their category, and by $\mathrm{SmProj}(\mathbb{Q})$ the full subcategory of smooth projective varieties over $\mathbb{Q}$.

One expects the existence of an abelian category $\mathrm{MM} = \mathrm{MM}(\mathbb{Q})_{\mathbb{Q}}$ of *mixed motives* (over $\mathbb{Q}$, with rational coefficients), and of a functor

$$h : \mathrm{Var}(\mathbb{Q}) \to \mathrm{MM}$$

which plays the role of universal cohomology. The morphisms in MM should be related to algebraic correspondences. In particular, the full subcategory NM of MM consisting of semisimple objects[5] has a simple description in terms of enumerative projective geometry: up to inessential technical modifications (idempotent completion, and inversion of the reduced motive $\mathbb{Q}(-1)$ of the projective line[6]), its objects are smooth projective varieties over $\mathbb{Q}$, its morphisms are given by algebraic correspondences up to numerical equivalence[7]. The restriction of h to $\mathrm{SmProj}(\mathbb{Q})$ takes values in NM.

In addition, the cartesian product on $\mathrm{Var}(\mathbb{Q})$ corresponds via h to a certain tensor product $\otimes$ on MM, which makes MM into a *tannakian category*[8].

[4] depending on the chosen version… In any case, the solution to our basic question in the case of period will rely on a transcendence conjecture of Grothendieck, which lies beyond foundational questions about motives.

[5] the so-called pure or numerical motives.

[6] which corresponds to inverting $2\pi i$ at the level of periods.

[7] Jannsen has proven that this category is indeed semisimple.

[8] which means, heuristically, that it has the same formal properties as the category of representations of a group.

The cohomologies H_{DR} and H_{B} factor through h, giving rise to two $\otimes$-functors

$$H_{\mathrm{DR}},\ H_{\mathrm{B}} : \mathrm{MM} \to \mathrm{Vec}_{\mathbb{Q}}$$

with values in the category of finite-dimensional $\mathbb{Q}$-vector spaces. Moreover, corresponding to the period isomorphism, one has an isomorphism of the complexified $\otimes$-functors (with values in $\mathrm{Vec}_{\mathbb{C}}$):

$$\varpi : H_{\mathrm{DR}} \otimes \mathbb{C} \cong H_{\mathrm{B}} \otimes \mathbb{C}.$$

In other words, there is a isomorphism in $\mathrm{Vec}_{\mathbb{C}}$

$$\varpi_M : H_{\mathrm{DR}}(M) \otimes \mathbb{C} \cong H_{\mathrm{B}}(M) \otimes \mathbb{C}$$

$\otimes$-functorial in the motive M. The entries of a matrix of ϖ_M with respect to some basis of the $\mathbb{Q}$-vector space $H_{\mathrm{DR}}(M)$ (*resp.* $H_B(M)$) are the *periods* of M.

3.4 Motivic Galois groups

Here comes the first fruit of this construction. Let $\langle M \rangle$ be the tannakian subcategory of MM generated by a motive M: its objets are given by algebraic constructions on M (sums, subquotients, duals, tensor products).

One defines the *motivic Galois group* of M to be the group-scheme

$$G_{\mathrm{mot}}(M) := \mathrm{Aut}^{\otimes} H_{\mathrm{B}|\langle M \rangle}$$

of automorphisms of the restriction of the $\otimes$-functor H_{B} to $\langle M \rangle$.

This is a linear algebraic group over $\mathbb{Q}$: in heuristic terms, $G_{\mathrm{mot}}(M)$ is just the Zariski-closed subgroup of $\mathrm{GL}(H_{\mathrm{B}}(M))$ consisting of matrices which preserve motivic relations in the algebraic constructions on $H_{\mathrm{B}}(M)$.

If $M = h(X)$ for some $X \in \mathrm{SmProj}(\mathbb{Q})$, it has the following concrete description: by Künneth formula and Poincaré duality, algebraic constructions on $H_{\mathrm{B}}(M)$ can be interpreted (up to Tate twists) as cohomology spaces for powers of X, and cohomology classes of algebraic cycles as certain mixed tensors on $H_{\mathrm{B}}(M)$. The motivic Galois group of X (or of M) is the closed subgroup of $\mathrm{GL}(H_{\mathrm{B}}(X))$ which fixes all cohomology classes of algebraic cycles on powers of X (interpreted as tensors).

3.5 Period torsors

Similarly, one can consider both H_{DR} and H_{B}, and define the *period torsor* of M to be the scheme

$$P_{\mathrm{mot}}(M) := \mathrm{Isom}^{\otimes}\,(H_{\mathrm{DR}|\langle M \rangle}, H_{\mathrm{B}|\langle M \rangle}) \in \mathrm{Var}(\mathbb{Q})$$

of isomorphisms of the restrictions of the $\otimes$-functors H_{DR} and H_{B} to $\langle M \rangle$. This is a torsor under $G_{\mathrm{mot}}(M)$, and it has a canonical complex point:

$$\varpi_M \in P_{\mathrm{mot}}(M)(\mathbb{C}).$$

3.6 Examples

(1) Let $F/\mathbb{Q}$ be a finite Galois extension contained in $\mathbb{C}$, and take $M = h(\operatorname{Spec} F)$. Then $G_{\mathrm{mot}}(M)$ is $\mathrm{Gal}(F/\mathbb{Q})$ viewed as a constant group-scheme over $\mathbb{Q}$, $P_{\mathrm{mot}}(M) = \operatorname{Spec} F$ and $\varpi_M \in P_{\mathrm{mot}}(M)(\mathbb{C}) = \mathrm{Hom}(F, \mathbb{C})$ is the canonical element.

(2) Let P be a projective space of dimension n, and $M = h(P)$. Then M decomposes as

$$\bigoplus_{i=0}^{i=n} h^{2i}(P), \quad h^{2i}(P) = \mathbb{Q}(-1)^{\otimes i}$$

where $\mathbb{Q}(-1)$ is the so-called Lefschetz motive. Then $G_{\mathrm{mot}}(M) = P_{\mathrm{mot}}(M) = \mathbb{G}_m$ (the multiplicative group), and $\varpi_M \in P_{\mathrm{mot}}(M)(\mathbb{C}) = \mathbb{C}^\times$ is $2\pi i$ (the period of $\mathbb{Q}(-1)$).

(3) Let E be an elliptic curve over $\mathbb{Q}$. Then $M = h(E)$ is an exterior algebra

$$\bigwedge h^1(E) = \bigoplus_{i=0}^{i=2} h^i(E), \quad h^2(E) = \bigwedge^2 h^1(E) = \mathbb{Q}(-1).$$

In the general (non CM) case, $G_{\mathrm{mot}}(M) = \mathrm{GL}(H^1_{\mathrm{B}}(E)) \cong \mathrm{GL}_{2\mathbb{Q}}$. In the CM case, there are non-trivial algebraic cycles on powers of E, and $G_{\mathrm{mot}}(M)$ is the normalizer of a Cartan subgroup of $\mathrm{GL}(H^1_{\mathrm{B}}(E))$ (*cf.* 2.2).

4 Grothendieck's period conjecture

4.1 Statement

Recall that for any motive M, the period torsor $P_{\mathrm{mot}}(M)$ is endowed with a canonical complex point

$$\varpi_M : \operatorname{Spec} \mathbb{C} \to P_{\mathrm{mot}}(M).$$

Conjecture (Grothendieck). *This is a **generic point**, i.e. the image of ϖ_M is the generic point of $P_{\mathrm{mot}}(M)$. Equivalently, the smallest algebraic subvariety of $P_{\mathrm{mot}}(M)$ defined over $\mathbb{Q}$ and containing ϖ is $P_{\mathrm{mot}}(M)$ itself.*

In more heuristic terms, this means that any polynomial relations with rational coefficients between periods should be of motivic origin (the relations of motivic origin being precisely those which define $P_{\mathrm{mot}}(M)$).

If $M = h(X)$ for some $X \in \mathrm{SmProj}(\mathbb{Q})$, the conjecture has the following concrete reformulation (it is stated in this way in [9]): by Künneth formula and Poincaré duality, cohomology classes of algebraic cycles can be viewed as certain mixed tensors on $H_{\mathrm{DR}}(X)$ and on $H_{\mathrm{B}}(X)$ respectively, which are compatible under ϖ_M. This gives rise to polynomial relations with rational coefficients between periods of X. Grothendieck's conjecture for X states that these relations generate the ideal of polynomial relations with rational coefficients between periods of X.

Here is a quantitative reformulation of the conjecture. Recall that the transcendence degree of a $\mathbb{Q}$-algebra is the maximal number of algebraically independent elements, or equivalently, the dimension of its spectrum. *Grothendieck's period conjecture for a motive M is equivalent to the conjunction of the following conditions:*

- $P_{\mathrm{mot}}(M)$ *is connected* (*but not necessarily geometrically connected*)[9],
- $\mathrm{tr.\,deg.}\,\mathbb{Q}[\mathrm{periods}(M)] = \dim\ G_{\mathrm{mot}}(M)$.

(this is clear if one remarks that $\mathrm{tr.\,deg.}\,\mathbb{Q}[\mathrm{periods}(M)]$ is the dimension of the $\mathbb{Q}$-Zariski closure of ϖ_M in $P_{\mathrm{mot}}(M)$).

4.2 Examples

Let us examine this conjecture in the three examples of 3.6. In the case $M = \mathrm{Spec}\, F$ (ordinary Galois theory), it is trivially true. For the motive of a projective space, it amounts to the transcendence of π.

For the motive of an elliptic curve over $\mathbb{Q}$ (or $\bar{\mathbb{Q}}$), it is known that the period torsor is connected, and the conjecture amounts to

$\mathrm{tr.\,deg.}\,\mathbb{Q}[\mathrm{periods}(M)] = 2$ in the CM case (which is Chudnovsky's theorem),

$\mathrm{tr.\,deg.}\,\mathbb{Q}[\mathrm{periods}(M)] = 4$ in the general case (which is open)[10].

4.3 Evidence

… is meager: apart from these examples, there is a general result by G. Wüstholz, which says that *linear* relations with coefficients in $\bar{\mathbb{Q}}$ between periods of 1*-motives* (motives associated to varieties of dimension ≤ 1) are of motivic origin[11] – and that is essentially all one knows in the present state of transcendental number theory (*cf.* [10] for more detail).

The limitation to linear relations comes from the fact that the proof relies on some kind of analytic uniformization of 1-motives, and no substitute for uniformization is available for tensor products of 1-motives. On the other hand, in the function-field analogous world of Drinfeld modules and Anderson's t-motives, there is a large class – stable under $\otimes$ – of objects which are uniformizable. This allows to obtain much stronger results in the direction of a function-field analog of Grothendieck's period conjecture, *cf.* e.g. [1].

[9] This condition would follow from standard Galois theory if, as it is expected, any motive with finite motivic Galois group comes from a finite extension of $\mathbb{Q}$.

[10] Only the inequality ≥ 2 is known.

[11] The standard way of stating the result is to say that linear relations with coefficients in $\bar{\mathbb{Q}}$ between periods of commutative algebraic groups over $\bar{\mathbb{Q}}$ come from endomorphisms.

Another heuristic justification comes from the parallel with other famous motivic conjecture such as the Hodge and Tate conjectures. Indeed, let $\mathcal{T}$ be the tannakian category whose objets consist in triples (V, W, ϖ), where $V, W \in \mathrm{Vec}_{\mathbb{Q}}$ and ϖ is an isomorphism $V_{\mathbb{C}} \cong W_{\mathbb{C}}$. One has a $\otimes$-functor, the *period realization*:

$$\mathrm{MM} \to \mathcal{T} : M \mapsto (H_{\mathrm{DR}}(M), H_{\mathrm{B}}(M), \varpi_M)$$

and Grothendieck's conjecture implies that this functor is fully faithful[12]. This is similar to the Hodge conjecture which, in Grothendieck's motivic formulation, asserts that the Hodge realization which maps to any mixed motive M over $\mathbb{C}$ the space $H_{\mathrm{B}}(M)$ endowed with its Hodge structure is fully faithful. The principle is the same: the realization, which is a rather plain linear structure, should nevertheless "capture" the algebro-geometric entity.

4.4 Kontsevich's viewpoint

By definition, periods are convergent integrals $\int_{\Sigma} \Omega$ of a certain type. They can be transformed by algebraic changes of variable, or using additivity of the integral, or using Stokes formula.

Kontsevich has conjectured that *any polynomial relation with rational coefficients between periods can be obtained by way of these elementary operations from calculus* (*cf.* [8]). Using ideas of Nori and the expected equivalence of various motivic settings, it can be shown that this conjecture is actually equivalent to Grothendieck's conjecture (*cf.* [3, chap. 23]).

5 Galois theory of periods

5.1 Setting

We come back to our basic question, in the case of periods.

Let α be a period. There exists a motive $M \in \mathrm{MM}$ such that $\alpha \in \mathbb{Q}[\mathrm{periods}(M)]$. Let us assume Grothendieck's period conjecture for M. Then $\mathbb{Q}[\mathrm{periods}(M)]$ coincides with the algebra $\mathbb{Q}[P_{\mathrm{mot}}(M)]$ of functions on $P_{\mathrm{mot}}(M)$. Since $P_{\mathrm{mot}}(M)$ is a torsor under $G_{\mathrm{mot}}(M)$, the group of rational points $G_{\mathrm{mot}}(M)(\mathbb{Q})$ acts on $\mathbb{Q}[P_{\mathrm{mot}}(M)]$, hence on $\mathbb{Q}[\mathrm{periods}(M)]$.

One may now define the *conjugates of* α to be the elements of the orbit $G_{\mathrm{mot}}(M)(\mathbb{Q}).\alpha$. It follows from Grothendieck's conjecture that this does not depend on M.

The *Galois closure* $\mathbb{Q}[\alpha]_{\mathrm{gal}}$ of $\mathbb{Q}[\alpha]$ is the subalgebra $\mathbb{Q}[G_{\mathrm{mot}}(M)(\mathbb{Q}).\alpha]$ of $\mathbb{Q}[\mathrm{periods}(M)]$.

[12] This is a weaker statement: for the tannakian category generated by a non-CM elliptic curve, it can be proven, whereas Grothendieck's conjecture itself is not known.

The *Galois group* of α is the smallest quotient G_α of $G_{\mathrm{mot}}(M)(\mathbb{Q})$ which acts effectively on $\mathbb{Q}[\alpha]_{\mathrm{gal}}$. Note that G_α acts transitively on the set of its conjugates and $\mathbb{Q}[\alpha]_{\mathrm{gal}}^{G_\alpha} = \mathbb{Q}$ (since $\mathbb{Q}[\mathrm{periods}(M)]^{G_{\mathrm{mot}}(M)(\mathbb{Q})} = \mathbb{Q}$).

Let us illustrate these definitions with a few examples.

5.2 Algebraic numbers

If α is an algebraic number, it follows from example 3.6 (1) that one recovers the usual notions of Galois theory.

5.3 The number π

It follows from example 3.6 (2) that one recovers the tentative answers of 2.1.

5.4 Elliptic periods

It follows from example 3.6 (3) that one recovers[13] the tentative answers of 2.2.

5.5 Special values of Γ

The special values of Euler's Gamma function at rationals $\frac{p}{q} \notin -\mathbb{N}$ are close to be periods: $\Gamma(\frac{p}{q})^q$ is a period of an abelian variety with complex multiplication by some cyclotomic field, and conversely, any period of such an abelian variety can be expressed as a polynomial in special values of Γ at rationals[14]. Grothendieck's conjecture for these abelian varieties amounts to say that any polynomial relation with rational coefficients between such numbers comes from the functional equations of Γ (Lang–Rohrlich conjecture). The structure of the corresponding motivic Galois groups is known (their connected parts are tori with explicit character groups), and it is possible in principle to describe the conjugates of $\Gamma(\frac{p}{q})$.

5.6 Logarithms

Let $\alpha = \log q$ with $q \in \mathbb{Q}\setminus\{-1, 0, 1\}$. This is a period of a so-called Kummer 1-motive M. Grothendieck's conjecture for M would imply that α and π are algebraically independent. If so, the conjugates of α are $\alpha + r\pi i$, $r \in \mathbb{Q}$ and G_α is a semi-direct product of $\mathbb{Q}^\times$ by $\mathbb{Q}$.

[13] In the non-CM case, one has to *assume* Grothendieck's conjecture for E, or at least that ω_1 and ω_2 are algebraically independent.

[14] by work of Shimura, Gross, Deligne, Anderson and others, *cf.* e.g. [3, chap. 24].

5.7 Zeta values

Let s be an odd integer > 1. Then $\alpha := \zeta(s) = \sum n^{-s}$ is a period of a so-called mixed Tate motive over $\mathbb{Z}$ (an extension of the unit motive by $\mathbb{Q}(s) = \mathbb{Q}(1)^{\otimes s}$). Grothendieck's conjecture for this type of motives would imply that π and $\zeta(3)$, $\zeta(5)$,... are algebraically independent, that the conjugates of $\zeta(s)$ are $\zeta(s) + r(\pi i)^s$, $r \in \mathbb{Q}$, and that G_α is a semi-direct product of $\mathbb{Q}^\times$ by $\mathbb{Q}$ (*cf.* [3, chap. 25]).

5.8 Multiple zeta values

More generally, multiple zeta values

$$\zeta(\underline{s}) = \sum_{n_1 > \ldots > n_k} n_1^{-s_1} \ldots n_k^{-s_k}$$

occur as periods of mixed Tate motives over $\mathbb{Z}$ (*cf.* e.g. [7] and [3, chap. 25]). Let us set

$$\mathfrak{z}_s = \sum_{s_1 + \ldots + s_k = s} \mathbb{Q}.\zeta(\underline{s}), \quad \mathfrak{z}_0 = \mathbb{Q}, \quad \mathfrak{z}_1 = 0.$$

Numerous relations between these periods have been discovered since Euler's times. For instance, $\sum \mathfrak{z}_s$ is a $\mathbb{Q}$-subalgebra of $\mathbb{R}$.

It is conjectured that the motivic Galois group corresponding to $\sum \mathfrak{z}_s$ is an extension of $\mathbb{Q}^\times$ by a prounipotent group whose Lie algebra, graded by the $\mathbb{Q}^\times$-action, is free with one generator in each odd degree $s > 1$. In any case, this group controls the relations between multiple zeta values, and using it, A. Goncharov and T. Terasoma have independently shown, unconditionally, that

$$\dim_{\mathbb{Q}} \mathfrak{z}_s \leq d_s$$

where d_s are the Taylor coefficients of $(1 - x^2 - x^3)^{-1}$.

On the other hand, it is expected that multiple zeta values are exactly the periods of mixed Tate motives over $\mathbb{Z}$ (Goncharov–Manin's conjecture). This combined with Grothendieck's period conjecture for these motives is equivalent to the conjecture that the sum $\sum \mathfrak{z}_s$ is direct (Hoffman's conjecture) and that $\dim_{\mathbb{Q}} \mathfrak{z}_s = d_s$ for any s (Zagier's conjecture).

Remarks. 1) The Galois theory of periods which we have outlined heavily relies upon Grothendieck's deep transcendence conjecture. However, one may hope that it could be useful for transcendental number theory: for instance, when trying to prove that a period α is transcendental, the a priori knowledge of its conjugates might be useful for the construction of auxiliary functions and other usual tools.

2) This is no relative version of this Galois theory, and only a partial Galois correspondence (between certain normal subgroups of Galois groups, and certain Galois-closed subalgebras of periods). Still, some twelve years ago, I proposed a generalized

period conjecture for periods of motives defined over non-algebraic fields, which contains both Grothendieck's conjecture and Schanuel's conjecture, *cf.* [3, chap. 23].

6 Relationship with differential Galois theory

Let us consider a smooth algebraic family $f : X \to S$. The variation of algebraic De Rham cohomology of the fibers X_s is controlled by a linear differential equation (Picard–Fuchs, or Gauss–Manin). More precisely, the periods ω_s of the fibers are multivalued analytic solutions of this differential equation.

The standard example, already known to Gauss, is the family of elliptic curves $y^2 = x(x-1)(x-s)$, whose periods are solutions of the hypergeometric differential equation with parameters $(\frac{1}{2}, \frac{1}{2}, 1)$ in the variable s.

Multivalued analytic solutions linear differential equations are subject to differential Galois theory: together with their derivatives, they generate the function algebra of a torsor under the corresponding differential Galois group. This is in particular the case for the functions ω_s.

Assume that f is defined over $\mathbb{Q}$ (or $\bar{\mathbb{Q}}$). Then for algebraic values σ of the parameter, the periods ω_σ of X_σ should be subject to a Galois theory related to motivic Galois theory, as outlined above.

Question. What about the relation between these two types of Galois theory, with respect to the specialization $s \mapsto \sigma$?

We shall sketch the answer in case f is *smooth projective* (in that case, it is indeed possible to prove an unconditional result, *cf.* [2, §5]).

Let $\mathcal{L}_{\mathrm{dif}}(s)$ denote the algebra of the differential Galois group of the Gauss–Manin connection attached to f, pointed at s. In fact, this connection is fuchsian, so that $\mathcal{L}_{\mathrm{dif}}(s)$ is just the Lie algebra of the complex Zariski closure of the monodromy group pointed at s in this case (Schlesinger's theorem). By Hodge–Deligne theory, it follows that when s varies, $(\mathcal{L}_{\mathrm{dif}}(s))$ form a local system of *semisimple* Lie algebras on S.

Let $\mathcal{L}_{\mathrm{mot}}(s)$ denote the Lie algebra of the (complexified) motivic Galois group of X_s. Since X_s is smooth projective, this is a *reductive* Lie algebra (whose dimension may vary with s).

Then *there is a local system* $(\mathcal{L}(s))$ *of reductive Lie subalgebras of* End $H_{\mathrm{B}}(X_s) \otimes \mathbb{C}$ *such that:*

a) *for any* $s \in S$, $\mathcal{L}_{\mathrm{dif}}(s)$ *is a Lie ideal of* $\mathcal{L}(s)$,

b) *for any* $s \in S$, $\mathcal{L}_{\mathrm{mot}}(s)$ *is a Lie subalgebra of* $\mathcal{L}(s)$,

c) *for any* s *outside some meager space of* $S(\mathbb{C})$, $\mathcal{L}_{\mathrm{mot}}(s) = \mathcal{L}(s)$,

d) *there are infinitely many* $\sigma \in S(\bar{\mathbb{Q}})$ *for which* $\mathcal{L}_{\mathrm{mot}}(\sigma) = \mathcal{L}(\sigma)$.

In the above example (family of elliptic curves), $\mathcal{L}_{\mathrm{dif}}(s) \cong sl_2$, $\mathcal{L}(s) \cong gl_2$, and $\mathcal{L}_{\mathrm{mot}}(\sigma) = \mathcal{L}(\sigma)$ except in the CM case.

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The combinatorics of Bogoliubov's recursion in renormalization

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Abstract. We describe various combinatorial aspects of the Birkhoff–Connes–Kreimer factorization in perturbative renormalization. The analog of Bogoliubov's preparation map on the Lie algebra of Feynman graphs is identified with the pre-Lie Magnus expansion. Our results apply to any connected filtered Hopf algebra, based on the pro-nilpotency of the Lie algebra of infinitesimal characters.

1 Introduction

The recursive nature of the process of renormalization in perturbative quantum field theory (with respect to the loop number of Feynman diagrams considered) has been settled long ago by N. N. Bogoliubov and O. S. Parasiuk [3], [26], [49]. However, its mathematical structure has become much more transparent since the seminal discovery of a Hopf algebra structure on Feynman diagrams by D. Kreimer [28] and subsequent work by A. Connes and Kreimer [7], [8], [9].

We present here a concise survey of several recent works on algebro-combinatorial aspects of the process of perturbative renormalization, in particular Bogoliubov's recursion respectively Connes–Kreimer's Birkhoff decomposition. The natural mathematical setting for such studies is provided by connected filtered Hopf algebras. Indeed, this leads, with the connectedness property being the very key to a recursive approach, to abstract versions of the counterterm character in general Rota–Baxter and dendriform algebras. Our approach allows us to understand the recursive nature of renormalization on the level of the pro-nilpotent Lie algebra of Feynman graphs. It turns out that the Baker–Campbell–Hausdorff recursion respectively the pre-Lie Magnus expansion provide the Lie algebraic analog of Bogoliubov's preparation map for Feynman graphs. We finish this survey by giving a short account of a natural matrix setting for perturbative renormalization which is well-suited for low-order explicit computations. Moreover, it allows for the transparent realization of the aforementioned abstract findings.

Let us briefly outline the organization of this survey. In the next section we review the essential notions from Hopf algebra theory. This section finishes with an abstract review on Connes–Kreimer's Birkhoff decomposition of Feynman rules in terms of Bogoliubov's preparation map. Section 2.6 contains an approach to Connes–Kreimer's Birkhoff decomposition based on a recursion defined in terms of the Baker–Campbell–Hausdorff formula. In Section 3 we analyze Bogoliubov's recursion from the point of view of Rota–Baxter algebras. These algebras provide the natural tools to understand Connes–Kreimer's finding of a factorization solved by Bogoliubov's formula. As it turns out, Loday's dendriform algebras serve as an abstract algebraic frame for one of the main aspects, i.e., iteration of Rota–Baxter maps and a particular interplay between an associative and a pre-Lie product induced by the Rota–Baxter structure. We finish Section 3 by dwelling on aspects related to renormalization theory. Finally, we briefly mention a non-Hopfian approach to Connes–Kreimer's finding in terms of triangular matrices providing a simple and straightforward setting for renormalization.

2 A summary of Birkhoff–Connes–Kreimer factorization

We introduce the crucial property of connectedness for bialgebras. The main interest resides in the possibility to implement recursive procedures in connected bialgebras, the induction taking place with respect to a filtration (e.g. the coradical filtration) or a grading. An important example of these techniques is the recursive construction of the antipode, which then "comes for free", showing that any connected bialgebra is in fact a connected Hopf algebra. The recursive nature of Bogoliubov's formula in the BPHZ [3], [26], [49] approach to perturbative renormalization ultimately comes from the connectedness of the underlying Hopf algebra respectively the corresponding pro-nilpotency of the Lie algebra of infinitesimal characters.

For details on bialgebras and Hopf algebras we refer the reader to the standard references, e.g. [46]. The use of bialgebras and Hopf algebras in combinatorics can at least be traced back to the seminal work of Joni and Rota [27].

2.1 Connected graded bialgebras

Let k be a field with characteristic zero. A *graded Hopf algebra* on k is a graded k-vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n,$$

endowed with a product $m\colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, a coproduct $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a unit $u\colon k \to \mathcal{H}$, a co-unit $\varepsilon\colon \mathcal{H} \to k$ and an antipode $S\colon \mathcal{H} \to \mathcal{H}$ fulfilling the usual axioms of a Hopf algebra [46], and such that

$$\begin{aligned} m(\mathcal{H}_p \otimes \mathcal{H}_q) &\subset \mathcal{H}_{p+q}, \\ \Delta(\mathcal{H}_n) &\subset \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q, \\ S(\mathcal{H}_n) &\subset \mathcal{H}_n. \end{aligned}$$

If we do not ask for the existence of an antipode S on $\mathcal{H}$ we get the definition of a *graded bialgebra*. In a graded bialgebra $\mathcal{H}$ we shall consider the increasing filtration

$$\mathcal{H}^n = \bigoplus_{p=0}^{n} \mathcal{H}_p.$$

Suppose moreover that $\mathcal{H}$ is *connected*, i.e., $\mathcal{H}_0$ is one-dimensional. Then we have

$$\operatorname{Ker} \varepsilon = \bigoplus_{n \geq 1} \mathcal{H}_n.$$

Proposition 2.1. *For any $x \in \mathcal{H}^n$, $n \geq 1$, we can write*

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \widetilde{\Delta} x, \quad \widetilde{\Delta} x \in \bigoplus_{\substack{p+q=n, \\ p \neq 0, q \neq 0}} \mathcal{H}_p \otimes \mathcal{H}_q.$$

The map $\widetilde{\Delta}$ is coassociative on $\operatorname{Ker} \varepsilon$ *and $\widetilde{\Delta}_k := (I^{\otimes k-1} \otimes \widetilde{\Delta})(I^{\otimes k-2} \otimes \widetilde{\Delta}) \ldots \widetilde{\Delta}$ sends $\mathcal{H}^n$ into $(\mathcal{H}^{n-k})^{\otimes k+1}$.*

Proof. Thanks to connectedness we clearly can write

$$\Delta x = a(x \otimes \mathbf{1}) + b(\mathbf{1} \otimes x) + \widetilde{\Delta} x,$$

with $a, b \in k$ and $\widetilde{\Delta} x \in \operatorname{Ker} \varepsilon \otimes \operatorname{Ker} \varepsilon$. The co-unit property then tells us that, with $k \otimes \mathcal{H}$ and $\mathcal{H} \otimes k$ canonically identified with $\mathcal{H}$,

$$x = (\varepsilon \otimes I)(\Delta x) = bx, \quad x = (I \otimes \varepsilon)(\Delta x) = ax,$$

hence $a = b = 1$. We shall use the following two variants of Sweedler's notation:

$$\Delta x = \sum_{(x)} x_1 \otimes x_2, \quad \widetilde{\Delta} x = \sum_{(x)} x' \otimes x'',$$

the second being relevant only for $x \in \operatorname{Ker} \varepsilon$. If x is homogeneous of degree n we can suppose that the components x_1, x_2, x', and x'' in the expressions above are homogeneous as well, and we have then $|x_1| + |x_2| = n$ and $|x'| + |x''| = n$. We easily compute

$$\begin{aligned}(\Delta \otimes I)\Delta(x) = & x \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes x \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{1} \otimes x \\ & + \sum_{(x)} x' \otimes x'' \otimes \mathbf{1} + x' \otimes \mathbf{1} \otimes x'' + \mathbf{1} \otimes x' \otimes x'' \\ & + (\widetilde{\Delta} \otimes I)\widetilde{\Delta}(x)\end{aligned}$$

and

$$\begin{aligned}(I \otimes \Delta)\Delta(x) = & x \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes x \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes x \\ & + \sum_{(x)} x' \otimes x'' \otimes \mathbf{1} + x' \otimes \mathbf{1} \otimes x'' + \mathbf{1} \otimes x' \otimes x'' \\ & + (I \otimes \widetilde{\Delta})\widetilde{\Delta}(x),\end{aligned}$$

hence the co-associativity of $\widetilde{\Delta}$ comes from the one of Δ. Finally it is easily seen by induction on k that for any $x \in \mathcal{H}^n$ we can write

$$\widetilde{\Delta}_k(x) = \sum_x x^{(1)} \otimes \cdots \otimes x^{(k+1)},$$

with $|x^{(j)}| \geq 1$. The grading imposes

$$\sum_{j=1}^{k+1} |x^{(j)}| = n,$$

so the maximum possible for any degree $|x^{(j)}|$ is $n - k$. □

2.2 Connected filtered bialgebras

A *filtered Hopf algebra* on k is a k-vector space together with an increasing $\mathbb{Z}_+$-indexed filtration

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n \subset \cdots, \quad \bigcup_n \mathcal{H}^n = \mathcal{H},$$

endowed with a product $m\colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, a coproduct $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a unit $u\colon k \to \mathcal{H}$, a co-unit $\varepsilon\colon \mathcal{H} \to k$ and an antipode $S\colon \mathcal{H} \to \mathcal{H}$ fulfilling the usual axioms of a Hopf algebra, and such that

$$m(\mathcal{H}^p \otimes \mathcal{H}^q) \subset \mathcal{H}^{p+q}, \qquad \Delta(\mathcal{H}^n) \subset \sum_{p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q, \quad \text{and} \quad S(\mathcal{H}^n) \subset \mathcal{H}^n.$$

If we do not ask for the existence of an antipode S on $\mathcal{H}$ we get the definition of a *filtered bialgebra*. For any $x \in \mathcal{H}$ we set

$$|x| := \min\{n \in \mathbb{N},\ x \in \mathcal{H}^n\}.$$

Any graded bialgebra or Hopf algebra is obviously filtered by the canonical filtration associated to the grading

$$\mathcal{H}^n := \bigoplus_{i=0}^{n} \mathcal{H}_i,$$

and in that case, if x is an homogeneous element, x is of degree n if and only if $|x| = n$. We say that the filtered bialgebra $\mathcal{H}$ is connected if $\mathcal{H}^0$ is one-dimensional. There is an analogue of Proposition 2.1 in the connected filtered case, the proof of which is very similar:

Proposition 2.2. *For any $x \in \mathcal{H}^n$, $n \geq 1$, we can write*

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \tilde{\Delta} x, \quad \tilde{\Delta} x \in \sum_{\substack{p+q=n,\\ p\neq 0, q\neq 0}} \mathcal{H}^p \otimes \mathcal{H}^q.$$

The map $\tilde{\Delta}$ is coassociative on $\operatorname{Ker}\varepsilon$ *and $\tilde{\Delta}_k = (I^{\otimes k-1} \otimes \tilde{\Delta})(I^{\otimes k-2} \otimes \tilde{\Delta}) \ldots \tilde{\Delta}$ sends $\mathcal{H}^n$ into $(\mathcal{H}^{n-k})^{\otimes k+1}$.*

The coradical filtration endows any pointed Hopf algebra $\mathcal{H}$ with a structure of filtered Hopf algebra (S. Montgomery, [37], Lemma 1.1). If $\mathcal{H}$ is moreover irreducible (i.e., if the image of k under the unit map u is the unique one-dimensional simple subcoalgebra of $\mathcal{H}$) this filtered Hopf algebra is moreover connected.

2.3 The convolution product

An important result is that any connected filtered bialgebra is indeed a filtered Hopf algebra, in the sense that the antipode comes for free. We give a proof of this fact as well as a recursive formula for the antipode with the help of the *convolution product*: let $\mathcal{H}$ be a (connected filtered) bialgebra, and let $\mathcal{A}$ be any k-algebra (which will be called the *target algebra*). The convolution product on the space $\mathcal{L}(\mathcal{H}, \mathcal{A})$ of linear maps from $\mathcal{H}$ to $\mathcal{A}$ is given by

$$\varphi * \psi(x) = m_{\mathcal{A}}(\varphi \otimes \psi)\Delta(x) = \sum_{(x)} \varphi(x_1)\psi(x_2).$$

Proposition 2.3. *The map $e = u_{\mathcal{A}} \circ \varepsilon$, given by $e(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ and $e(x) = 0$ for any $x \in$* $\operatorname{Ker}\varepsilon$*, is a unit for the convolution product. Moreover the set $G(\mathcal{A}) := \{\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A}),\ \varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}\}$ endowed with the convolution product is a group.*

Proof. The first statement is straightforward. To prove the second let us consider the formal series

$$\varphi^{*-1}(x) = \big(e - (e - \varphi)\big)^{*-1}(x) = \sum_{m \geq 0} (e - \varphi)^{*m}(x).$$

Using $(e-\varphi)(\mathbf{1}) = 0$ we have immediately $(e-\varphi)^{*m}(\mathbf{1}) = 0$, and for any $x \in \operatorname{Ker}\varepsilon$

$$(e-\varphi)^{*n}(x) = m_{\mathcal{A},n-1}(\underbrace{\varphi \otimes \cdots \otimes \varphi}_{n \text{ times}})\widetilde{\Delta}_{n-1}(x).$$

When $x \in \mathcal{H}^p$ this expression vanishes then for $n \geq p+1$. The formal series ends up then with a finite number of terms for any x, which proves the result. □

Corollary 2.4. *Any connected filtered bialgebra $\mathcal{H}$ is a filtered Hopf algebra. The antipode is defined by*

$$S(x) = \sum_{m \geq 0} (u \circ \varepsilon - I)^{*m}(x). \tag{2.1}$$

It is given by $S(\mathbf{1}) = \mathbf{1}$ and recursively by any of the two formulas for $x \in \operatorname{Ker}\varepsilon$:

$$S(x) = -x - \sum_{(x)} S(x')x'' \quad \textit{and} \quad S(x) = -x - \sum_{(x)} x'S(x'').$$

Proof. The antipode, when it exists, is the inverse of the identity for the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{H})$. One just needs then to apply Proposition 2.3 with $\mathcal{A} = \mathcal{H}$. The two recursive formulas follow directly from the two equalities

$$m(S \otimes I)\Delta(x) = 0 = m(I \otimes S)\Delta(x),$$

fulfilled by any $x \in \operatorname{Ker}\varepsilon$. □

Let $\mathfrak{g}(\mathcal{A})$ be the subspace of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ formed by the elements α such that $\alpha(\mathbf{1}) = 0$. It is clearly a subalgebra of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ for the convolution product. We have

$$G(\mathcal{A}) = e + \mathfrak{g}(\mathcal{A}). \tag{2.2}$$

From now on we shall suppose that the ground field k is of characteristic zero. For any $x \in \mathcal{H}^n$, the exponential

$$\exp^*(\alpha)(x) = \sum_{k \geq 0} \frac{\alpha^{*k}(x)}{k!}$$

is a finite sum (ending up at $k = n$). It is a bijection from $\mathfrak{g}(\mathcal{A})$ onto $G(\mathcal{A})$. Its inverse is given by

$$\log^*(e+\alpha)(x) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \alpha^{*k}(x).$$

This sum again ends up at $k = n$ for any $x \in \mathcal{H}^n$. Let us introduce a decreasing filtration on $\mathcal{L} = \mathcal{L}(\mathcal{H}, \mathcal{A})$:

$$\mathcal{L}^n := \{\alpha \in \mathcal{L},\ \alpha_{|_{\mathcal{H}^{n-1}}} = 0\}.$$

Clearly $\mathcal{L}^0 = \mathcal{L}$ and $\mathcal{L}^1 = \mathfrak{g}(\mathcal{A})$. We define the valuation $\operatorname{val}\varphi$ of an element φ of $\mathcal{L}$ as the biggest integer k such that φ is in $\mathcal{L}^k$. We shall consider in the sequel the ultrametric distance on $\mathcal{L}$ induced by the filtration

$$d(\varphi, \psi) = 2^{-\operatorname{val}(\varphi-\psi)}. \tag{2.3}$$

For any $\alpha, \beta \in \mathfrak{g}(\mathcal{A})$ let $[\alpha, \beta] = \alpha * \beta - \beta * \alpha$.

Proposition 2.5. *We have the inclusion*

$$\mathcal{L}^p * \mathcal{L}^q \subset \mathcal{L}^{p+q},$$

and moreover the metric space $\mathcal{L}$ endowed with the distance defined by (2.3) *is complete.*

Proof. Take any $x \in \mathcal{H}^{p+q-1}$, and any $\alpha \in \mathcal{L}^p$ and $\beta \in \mathcal{L}^q$. We have

$$(\alpha * \beta)(x) = \sum_{(x)} \alpha(x_1)\beta(x_2).$$

Recall that we denote by $|x|$ the minimal n such that $x \in \mathcal{H}^n$. Since $|x_1| + |x_2| = |x| \leq p+q-1$, either $|x_1| \leq p-1$ or $|x_2| \leq q-1$, so the expression vanishes. Now if (ψ_n) is a Cauchy sequence in $\mathcal{L}$ it is immediate to see that this sequence is *locally stationary*, i.e., for any $x \in \mathcal{H}$ there exists $N(x) \in \mathbb{N}$ such that $\psi_n(x) = \psi_{N(x)}(x)$ for any $n \geq N(x)$. Then the limit of (ψ_n) exists and is clearly defined by

$$\psi(x) = \psi_{N(x)}(x). \qquad \square$$

As a corollary the Lie algebra $\mathcal{L}^1 = \mathfrak{g}(\mathcal{A})$ is *pro-nilpotent*, in the sense that it is the projective limit of the Lie algebras $\mathfrak{g}(\mathcal{A})/\mathcal{L}^n$, which are nilpotent.

2.4 Characters and infinitesimal characters

Let $\mathcal{H}$ be a connected filtered Hopf algebra over k, and let $\mathcal{A}$ be a *commutative* k-algebra. We shall consider unital algebra morphisms from $\mathcal{H}$ to the target algebra $\mathcal{A}$, which we shall call slightly abusively *characters*. We recover of course the usual notion of character when the algebra $\mathcal{A}$ is the ground field k. The notion of character involves only the algebra structure of $\mathcal{H}$. On the other hand the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{A})$ involves only the *coalgebra* structure on $\mathcal{H}$. Let us consider now the full Hopf algebra structure on $\mathcal{H}$ and see what happens to characters with the convolution product.

Proposition 2.6. *Let $\mathcal{H}$ be a connected filtered Hopf algebra over k, and let $\mathcal{A}$ be a commutative k-algebra. Then the characters from $\mathcal{H}$ to $\mathcal{A}$ form a group $G_1(\mathcal{A})$ under the convolution product, and for any $\varphi \in G_1(\mathcal{A})$ the inverse is given by*

$$\varphi^{*-1} = \varphi \circ S.$$

We call *infinitesimal characters with values in the algebra $\mathcal{A}$* those elements α of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ such that

$$\alpha(xy) = e(x)\alpha(y) + \alpha(x)e(y).$$

Proposition 2.7. *Let $G_1(\mathcal{A})$ (resp. $\mathfrak{g}_1(\mathcal{A})$) be the set of characters of $\mathcal{H}$ with values in $\mathcal{A}$ (resp. the set of infinitesimal characters of $\mathcal{H}$ with values in $\mathcal{A}$). Then $G_1(\mathcal{A})$ is a subgroup of G, the exponential restricts to a bijection from $\mathfrak{g}_1(\mathcal{A})$ onto $G_1(\mathcal{A})$, and $\mathfrak{g}_1(\mathcal{A})$ is a Lie subalgebra of $\mathfrak{g}(\mathcal{A})$.*

Proof. Part of these results is a reformulation of Proposition 2.6 and some points are straightforward. The only non-trivial point concerns $\mathfrak{g}_1(\mathcal{A})$ and $G_1(\mathcal{A})$. Take two infinitesimal characters α and β with values in $\mathcal{A}$ and compute:

$$\begin{aligned}(\alpha * \beta)(xy) &= \sum_{(x)(y)} \alpha(x_1y_1)\beta(x_2y_2)\\ &= \sum_{(x)(y)} \big(\alpha(x_1)e(y_1) + e(x_1)\alpha(y_1)\big).\big(\beta(x_2)e(y_2) + e(x_2)\alpha(y_2)\big)\\ &= (\alpha * \beta)(x)e(y) + \alpha(x)\beta(y) + \beta(x)\alpha(y) + e(x)(\alpha * \beta)(y).\end{aligned}$$

Using the commutativity of $\mathcal{A}$ we immediately get

$$[\alpha, \beta](xy) = [\alpha, \beta](x)e(y) + e(x)[\alpha, \beta](y),$$

which shows that $\mathfrak{g}_1(\mathcal{A})$ is a Lie algebra. Now for $\alpha \in \mathfrak{g}_1(\mathcal{A})$ we have

$$\alpha^{*n}(xy) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{*k}(x)\alpha^{*(n-k)}(y),$$

as is easily seen by induction on n. A straightforward computation then yields

$$\exp^*(\alpha)(xy) = \exp^*(\alpha)(x) \exp^*(\alpha)(y).$$ □

2.5 Renormalization in connected filtered Hopf algebras

We describe in this section the renormalization à la Connes–Kreimer ([28], [7], [8]) in the abstract context of connected filtered Hopf algebras: the objects to be renormalised are characters with values in a commutative unital target algebra $\mathcal{A}$ endowed with a *renormalization scheme*, i.e., a splitting $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ into two subalgebras with $\mathbf{1} \in \mathcal{A}_+$. An important example is given by the *minimal subtraction* (MS) *scheme* on the algebra $\mathcal{A}$ of meromorphic functions of one variable z, where $\mathcal{A}_+$ is the algebra of meromorphic functions which are holomorphic at $z = 0$, and where $\mathcal{A}_- = z^{-1}\mathbb{C}[z^{-1}]$ stands for the "polar parts". Any $\mathcal{A}$-valued character φ admits a unique *Birkhoff decomposition*

$$\varphi = \varphi_-^{*-1} * \varphi_+,$$

where φ_+ is an $\mathcal{A}_+$-valued character, and where $\varphi_-(\operatorname{Ker}\varepsilon) \subset \mathcal{A}_-$. In the MS scheme case described just above, the renormalised character is the scalar-valued character given by the evaluation of φ_+ at $z = 0$ (whereas the evaluation of φ at $z = 0$ does not necessarily make sense).

Theorem 2.8 (Factorization of the group $G(\mathcal{A})$). (1) *Let $\mathcal{H}$ be a connected filtered Hopf algebra. Let $\mathcal{A}$ be a commutative unital algebra with a renormalization scheme, and let $\pi : \mathcal{A} \to \mathcal{A}$ be the projection onto $\mathcal{A}_-$ parallel to $\mathcal{A}_+$. Let $G(\mathcal{A})$ be the group of those $\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ such that $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ endowed with the convolution product. Any $\varphi \in G(\mathcal{A})$ admits a unique Birkhoff decomposition*

$$\varphi = \varphi_-^{*-1} * \varphi_+, \tag{2.4}$$

where φ_- sends $\mathbf{1}$ to $\mathbf{1}_{\mathcal{A}}$ and $\operatorname{Ker} \varepsilon$ into $\mathcal{A}_-$, and where φ_+ sends $\mathcal{H}$ into $\mathcal{A}_+$. The maps φ_- and φ_+ are given on $\operatorname{Ker} \varepsilon$ by the following recursive formulas

$$\varphi_-(x) = -\pi\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big),$$

$$\varphi_+(x) = (I - \pi)\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big).$$

where I is the identity map.

(2) *If φ is a character, the components φ_- and φ_+ occurring in the Birkhoff decomposition of φ are characters as well.*

Proof. The proof goes along the same lines as the proof of Theorem 4 of [8]: for the first assertion it is immediate from the definition of π that φ_- sends $\operatorname{Ker} \varepsilon$ into $\mathcal{A}_-$, and that φ_+ sends $\operatorname{Ker} \varepsilon$ into $\mathcal{A}_+$. It only remains to check equality $\varphi_+ = \varphi_- * \varphi$, which is an easy computation

$$\begin{aligned}\varphi_+(x) &= (I - \pi)\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big). \\ &= \varphi(x) + \varphi_-(x) + \sum_{(x)} \varphi_-(x')\varphi(x'') \\ &= (\varphi_- * \varphi)(x).\end{aligned}$$

The proof of assertion (2) can be carried out exactly as in [8] and relies on the following *Rota–Baxter relation* in $\mathcal{A}$:

$$\pi(a)\pi(b) = \pi\big(\pi(a)b + a\pi(b)\big) - \pi(ab), \tag{2.5}$$

which is easily verified by decomposing a and b into their $\mathcal{A}_\pm$-parts. We will derive below a more conceptual proof. □

Remark 2.9. Define the Bogoliubov preparation map as the map $B : G(\mathcal{A}) \to \mathcal{L}(\mathcal{H}, \mathcal{A})$ given by

$$B(\varphi) = \varphi_- * (\varphi - e), \tag{2.6}$$

such that for any $x \in \operatorname{Ker} \varepsilon$ we have

$$B(\varphi)(x) = \varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'').$$

The components of φ in the Birkhoff decomposition read

$$\varphi_- = e - \pi \circ B(\varphi), \quad \varphi_+ = e + (I - \pi) \circ B(\varphi). \tag{2.7}$$

On $\operatorname{Ker}\varepsilon$ they reduce to $-\pi \circ B(\varphi)$, $(I-\pi)\circ B(\varphi)$, respectively. Plugging equation (2.6) inside (2.7) and setting $\alpha := e - \varphi$ we get the following expression for φ_-:

$$\begin{aligned} \varphi_- &= e + P(\varphi_- * \alpha) \\ &= e + P(\alpha) + P\big(P(\alpha) * \alpha\big) + \cdots + \underbrace{P\big(P\big(\ldots P(}_{n \text{ times}}\alpha) * \alpha\big) \cdots * \alpha\big) + \cdots \end{aligned} \tag{2.8}$$

and for φ_+ we find

$$\begin{aligned} \varphi_+ &= e - \widetilde{P}(\varphi_- * \alpha) \\ &= e - \widetilde{P}(\varphi_+ * \beta) \\ &= e - \widetilde{P}(\beta) + \widetilde{P}\big(\widetilde{P}(\beta) * \beta\big) - \cdots + (-1)^n \underbrace{\widetilde{P}\big(\widetilde{P}\big(\ldots \widetilde{P}(}_{n \text{ times}}\beta) * \beta\big) \cdots * \beta\big) + \cdots \end{aligned} \tag{2.9}$$

with $\beta := \varphi^{-1} * \alpha = \varphi^{-1} - e$, and where $\widetilde{P}$ and P are projections on $\mathcal{L}(\mathcal{H}, \mathcal{A})$ defined by $\widetilde{P}(\alpha) = (I - \pi) \circ \alpha$ and $P(\alpha) = \pi \circ \alpha$, respectively.

2.6 The Baker–Campbell–Hausdorff recursion

Let $\mathcal{L}$ be any complete filtered Lie algebra. Thus $\mathcal{L}$ has a decreasing filtration $(\mathcal{L}_n)$ of Lie subalgebras such that $[\mathcal{L}_m, \mathcal{L}_n] \subseteq \mathcal{L}_{m+n}$ and $\mathcal{L} \cong \varprojlim \mathcal{L}/\mathcal{L}_n$ (i.e., $\mathcal{L}$ is complete with respect to the topology induced by the filtration). Let A be the completion of the enveloping algebra $\mathcal{U}(\mathcal{L})$ for the decreasing filtration naturally coming from that of $\mathcal{L}$. The functions

$$\exp\colon A_1 \to 1 + A_1, \qquad \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

$$\log\colon 1 + A_1 \to A_1, \qquad \log(1+a) = -\sum_{n=1}^{\infty} \frac{(-a)^n}{n}$$

are well-defined and are the inverse of each other. The Baker–Campbell–Hausdorff (BCH) formula writes for any $x, y \in \mathcal{L}_1$ [41], [48] as follows:

$$\exp(x)\exp(y) = \exp\big(C(x,y)\big) = \exp\big(x + y + \mathrm{BCH}(x,y)\big),$$

where $\mathrm{BCH}(x,y)$ is an element of $\mathcal{L}_2$ given by a Lie series the first few terms of which are

$$\mathrm{BCH}(x,y) = \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[y,[y,x]] - \frac{1}{24}[x,[y,[x,y]]] + \cdots$$

Now let $P: \mathcal{L} \to \mathcal{L}$ be any linear map preserving the filtration of $\mathcal{L}$. We define $\widetilde{P}$ to be $\mathrm{Id}_{\mathcal{L}} - P$. For $a \in \mathcal{L}_1$, define $\chi(a) = \lim_{n\to\infty} \chi_{(n)}(a)$ where $\chi_{(n)}(a)$ is given by the BCH-recursion

$$\begin{aligned} \chi_{(0)}(a) &:= a, \\ \chi_{(n+1)}(a) &= a - \mathrm{BCH}\big(P(\chi_{(n)}(a)),\ (\mathrm{Id}_{\mathcal{L}} - P)(\chi_{(n)}(a))\big), \end{aligned} \tag{2.10}$$

and where the limit is taken with respect to the topology given by the filtration. Then the map $\chi: \mathcal{L}_1 \to \mathcal{L}_1$ satisfies

$$\chi(a) = a - \mathrm{BCH}\big(P(\chi(a)),\ \widetilde{P}(\chi(a))\big). \tag{2.11}$$

This map appeared in [13], [12], where more details can be found, see also [35], [36]. The following proposition ([16], [35]) gives further properties of the map χ.

Proposition 2.10. *For any linear map $P: \mathcal{L} \to \mathcal{L}$ preserving the filtration of $\mathcal{L}$ there exists a (usually non-linear) unique map $\chi: \mathcal{L}_1 \to \mathcal{L}_1$ such that $(\chi - \mathrm{Id}_{\mathcal{L}})(\mathcal{L}_i) \subset \mathcal{L}_{2i}$ for any $i \geq 1$, and such that, with $\widetilde{P} := \mathrm{Id}_{\mathcal{L}} - P$ we have*

$$a = C\big(P\big(\chi(a)\big),\ \widetilde{P}\big(\chi(a)\big)\big) \quad \textit{for all } a \in \mathcal{L}_1. \tag{2.12}$$

This map is bijective, and its inverse is given by

$$\chi^{-1}(a) = C\big(P(a),\ \widetilde{P}(a)\big) = a + \mathrm{BCH}\big(P(a),\ \widetilde{P}(a)\big). \tag{2.13}$$

Proof. Equation (2.12) can be rewritten as

$$\chi(a) = F_a\big(\chi(a)\big),$$

with $F_a: \mathcal{L}_1 \to \mathcal{L}_1$ defined by

$$F_a(b) = a - \mathrm{BCH}\big(P(b), \widetilde{P}(b)\big).$$

This map F_a is a contraction with respect to the metric associated with the filtration: indeed if $b, \varepsilon \in \mathcal{L}_1$ with $\varepsilon \in \mathcal{L}_n$, we have

$$F_a(b+\varepsilon) - F_a(b) = \mathrm{BCH}\big(P(b),\ \widetilde{P}(b)\big) - \mathrm{BCH}\big(P(b+\varepsilon),\ \widetilde{P}(b+\varepsilon)\big).$$

The right-hand side is a sum of iterated commutators in each of which ε does appear at least once. So it belongs to $\mathcal{L}_{n+1}$. So the sequence $F_a^n(b)$ converges in $\mathcal{L}_1$ to a unique fixed point $\chi(a)$ for F_a.

Let us remark that for any $a \in \mathcal{L}_i$, then, by a straightforward induction argument, $\chi_{(n)}(a) \in \mathcal{L}_i$ for any n, so $\chi(a) \in \mathcal{L}_i$ by taking the limit. Then the difference $\chi(a) - a = \mathrm{BCH}\big(P\big(\chi(a)\big),\ \widetilde{P}\big(\chi(a)\big)\big)$ clearly belongs to $\mathcal{L}_{2i}$. Now consider the map $\psi: \mathcal{L}_1 \to \mathcal{L}_1$ defined by $\psi(a) = C\big(P(a),\ \widetilde{P}(a)\big)$. It is clear from the definition of χ that $\psi \circ \chi = \mathrm{Id}_{\mathcal{L}_1}$. Then χ is injective and ψ is surjective. The injectivity of ψ will be an immediate consequence of the following lemma.

Lemma 2.11. *The map ψ increases the ultrametric distance given by the filtration.*

Proof. For any $x, y \in \mathcal{L}_1$ the distance $d(x, y)$ is given by 2^{-n} where $n = \sup\{k \in \mathbb{N},\ x - y \in \mathcal{L}_k\}$. We have then to prove that $\psi(x) - \psi(y) \notin \mathcal{L}_{n+1}$. But:

$$\begin{aligned}\psi(x) - \psi(y) &= x - y + \mathrm{BCH}\big(P(x),\ \widetilde{P}(x)\big) - \mathrm{BCH}\big(P(y),\ \widetilde{P}(y)\big)\\ &= x - y + \Big(\mathrm{BCH}\big(P(x),\ \widetilde{P}(x)\big)\\ &\qquad - \mathrm{BCH}\big(P(x) - P(x-y),\ \widetilde{P}(x) - \widetilde{P}(x-y)\big)\Big).\end{aligned}$$

The rightmost term inside the large brackets clearly belongs to $\mathcal{L}_{n+1}$. As $x-y \notin \mathcal{L}_{n+1}$ by hypothesis, this proves the claim. □

The map ψ is then a bijection, so χ is also bijective, which proves Proposition 2.10. □

Corollary 2.12. *For any $a \in \mathcal{L}_1$ we have the following equality taking place in $1 + A_1 \subset A$:*

$$\exp(a) = \exp\big(P(\chi(a))\big)\exp\big(\widetilde{P}(\chi(a))\big). \tag{2.14}$$

Putting (2.8) and (2.14) together we get for any $\alpha \in \mathcal{L}_1$ the following *non-commutative Spitzer identity*:

$$e+P(\alpha)+\cdots+\underbrace{P\big(P\big(\ldots P}_{n\text{ times}}(\alpha)*\alpha\big)\cdots*\alpha\big)+\cdots = \exp\big[-P\big(\chi\big(\log(e-\alpha)\big)\big)\big]. \tag{2.15}$$

This identity is valid for any filtration-preserving Rota–Baxter operator P in a complete filtered Lie algebra (see Section 3). For a detailed treatment of these aspects, see [13], [12], [16], [22].

Remark 2.13. Using (2.14) the reader should have no problem in verifying that the Baker–Campbell–Hausdorff recursion (2.11) can also be written more compactly:

$$\chi(a) = a + \mathrm{BCH}\big(-P(\chi(a)),\ a\big). \tag{2.16}$$

2.7 Application to perturbative renormalization I

Suppose now that $\mathcal{L} = \mathcal{L}(\mathcal{H}, \mathcal{A})$ (with the setup and notations of paragraph 2.5), and that the operator P is now the projection defined by $P(a) = \pi \circ a$. It is clear that Corollary 2.12 applies in this setting and that the first factor on the right-hand side of (2.14) is an element of $G_1(\mathcal{A})$, the group of $\mathcal{A}$-valued characters of $\mathcal{H}$, which sends $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_-$, and that the second factor is an element of G_1 which sends $\mathcal{H}$ into $\mathcal{A}_+$. Going back to Theorem 2.8 and using uniqueness of the decomposition (2.4) we then see that (2.14) in fact is the Birkhoff–Connes–Kreimer decomposition of the element $\exp^*(a)$ in G_1. Indeed, starting with the infinitesimal character a in the Lie algebra $\mathfrak{g}_1(\mathcal{A})$ equation (2.14) gives the Birkhoff–Connes–Kreimer decomposition of $\varphi = \exp^*(a)$ in the group $G_1(\mathcal{A})$ of $\mathcal{A}$-valued characters of $\mathcal{H}$, i.e.,

$$\varphi_- = \exp^*\big(-P(\chi(a))\big) \quad \text{and} \quad \varphi_+ = \exp^*\big(\widetilde{P}(\chi(a))\big) \quad \text{such that } \varphi = \varphi_-^{-1} * \varphi_+,$$

thus proving the second assertion in Theorem 2.8.

Therefore, we may say that the Baker–Campbell–Hausdorff recursion (2.16) encodes the process of renormalization on the level of the Lie algebra $\mathfrak{g}_1(\mathcal{A})$ of infinitesimal characters. Indeed, for $a \in \mathfrak{g}_1(\mathcal{A})$, determined by the Feynman rules character $\varphi = \exp^*(a)$, we calculate the element $b(a) := \chi(a)$ in $\mathfrak{g}_1(\mathcal{A})$, i.e., the Lie algebra analog of Bogoliubov's preparation map (2.6), such that

$$a = C\big(P(b(a)), \tilde{P}(b(a))\big) = C\big(P(\chi(a)), \tilde{P}(\chi(a)\big) \tag{2.17}$$

gives rise to (2.14). By construction, $P(b(a))$ and $\tilde{P}(b(a))$ take values in $\mathcal{A}_\mp$, respectively. Hence, the decomposition (2.17) of $a \in \mathfrak{g}_1(\mathcal{A})$ is the Lie algebra analog of the Birkhoff–Connes–Kreimer decomposition of $\varphi = \exp^*(a)$.

Comparing Corollary 2.12 and Theorem 2.8 the reader may wonder upon the role played by the Rota–Baxter relation (2.5) for the projector P. In the following section we will show that it is this identity that allows to write the exponential $\varphi_- = \exp^*\big(-P(\chi(a))\big)$ as a recursion, that is, $\varphi_- = e - P(B(\varphi))$, where $B(\varphi) = \varphi_- * (\varphi - e)$. Equivalently, this amounts to the fact that the group $G_1(\mathcal{A})$ factorizes into two subgroups $G_1^-(\mathcal{A})$ and $G_1^+(\mathcal{A})$, such that $\varphi_\pm \in G_1^\pm(\mathcal{A})$.

3 Rota–Baxter and dendriform algebras

We are interested in abstract versions of identities (2.8) and (2.15) fulfilled by the counterterm character φ_-. The general algebraic context is given by Rota–Baxter (associative) algebras of weight θ, which are themselves dendriform algebras. We first briefly recall the definition of Rota–Baxter (RB) algebra and its most important properties. For more details we refer the reader to the classical papers [1], [2], [5], [42], [43], as well as for instance to the references [15], [16].

Let A be an associative not necessarily unital nor commutative algebra with $R \in \mathrm{End}(A)$. The product of a and b in A is written $a \cdot b$ or simply ab when no confusion can arise. We call a tuple (A, R) a *Rota–Baxter algebra* of weight $\theta \in k$ if R satisfies the *Rota–Baxter relation*

$$R(x)R(y) = R\big(R(x)y + xR(y) + \theta xy\big). \tag{3.1}$$

Note that the operator P of paragraph 2.5 is an idempotent Rota–Baxter operator. Its weight is thus $\theta = -1$. Changing R to $R' := \mu R$, $\mu \in k$, gives rise to a RB algebra of weight $\theta' := \mu\theta$, so that a change in the θ parameter can always be achieved, at least as long as weight non-zero RB algebras are considered. The definition generalizes to other types of algebras than associative algebras: for example one may want to consider RB Lie or pre-Lie algebra structures. Further below we will encounter examples of such structures.

Let us recall some classical examples of RB algebras. First, consider the integration by parts rule for the Riemann integral map. Let $A := C(\mathbb{R})$ be the ring of real continuous functions with pointwise product. The indefinite Riemann integral can be seen as a linear map on A:

$$I: A \to A, \quad I(f)(x) := \int_0^x f(t)\,dt. \tag{3.2}$$

Then, integration by parts for the Riemann integral can be written compactly as

$$I(f)(x)I(g)(x) = I\big(I(f)g\big)(x) + I\big(fI(g)\big)(x), \tag{3.3}$$

dually to the classical Leibniz rule for derivations. Hence, we found our first example of a weight zero Rota–Baxter map. Correspondingly, on a suitable class of functions, we define the following Riemann summation operators:

$$R_\theta(f)(x) := \sum_{n=1}^{[x/\theta]} \theta f(n\theta) \quad \text{and} \quad R'_\theta(f)(x) := \sum_{n=1}^{[x/\theta]+1} \theta f(n\theta). \tag{3.4}$$

We observe readily that

$$\begin{aligned}
&\Big(\sum_{n=1}^{[x/\theta]} \theta f(n\theta)\Big)\Big(\sum_{m=1}^{[x/\theta]} \theta g(m\theta)\Big) \\
&\qquad = \Big(\sum_{n>m=1}^{[x/\theta]} + \sum_{m>n=1}^{[x/\theta]} + \sum_{m=n=1}^{[x/\theta]}\Big)\theta^2 f(n\theta)g(m\theta) \\
&\qquad = \sum_{m=1}^{[x/\theta]} \theta^2\Big(\sum_{k=1}^{m} f\big(k\theta\big)\Big)g(m\theta) \\
&\qquad\qquad + \sum_{n=1}^{[x/\theta]} \theta^2\Big(\sum_{k=1}^{n} g\big(k\theta\big)\Big)f(n\theta) - \sum_{n=1}^{[x/\theta]} \theta^2 f(n\theta)g(n\theta) \\
&\quad = R_\theta\big(R_\theta(f)g\big)(x) + R_\theta\big(fR_\theta(g)\big)(x) + \theta R_\theta(fg)(x).
\end{aligned} \tag{3.5}$$

Similarly for the map R'_θ except that the diagonal, counted twice, must be subtracted instead of added. Hence, the Riemann summation maps R_θ and R'_θ satisfy the weight θ and the weight $-\theta$ Rota–Baxter relation, respectively.

Proposition 3.1. *Let (A, R) be a Rota–Baxter algebra. The map $\widetilde{R} = -\theta\,\mathrm{id}_A - R$ is a Rota–Baxter map of weight θ on A. The images of R and $\widetilde{R}$, $A_\mp \subseteq A$, respectively are subalgebras in A.*

We omit the proof since it follows directly from the Rota–Baxter relation. A *Rota–Baxter ideal* of a Rota–Baxter algebra (A, R) is an ideal $I \subset A$ such that $R(I) \subseteq I$.

The Rota–Baxter relation extends naturally to the Lie algebra L_A corresponding to A:

$$[R(x), R(y)] = R\big([R(x), y] + [x, R(y)]\big) + \theta R\big([x, y]\big),$$

making (L_A, R) into a Rota–Baxter Lie algebra of weight θ.

Proposition 3.2. *The vector space underlying A equipped with the product*

$$x *_\theta y := R(x)y + xR(y) + \theta xy \tag{3.6}$$

is again a Rota–Baxter algebra of weight θ with Rota–Baxter map R. We denote it by (A_θ, R) and call it double Rota–Baxter algebra. The Rota–Baxter map R becomes a (not necessarily unital even if A is unital) algebra homomorphism from the algebra A_θ to A.

Let us remark that for the corresponding Lie algebra L_A we find the new Lie bracket (compare with [44])

$$[x, y]_\theta := [R(x), y] + [x, R(y)] + \theta[x, y].$$

One sees immediately that $x *_\theta y = -\theta^{-1}\big(R(x)R(y) - \tilde{R}(x)\tilde{R}(y)\big)$, and

$$R\big(a *_\theta b\big) = R(a)R(b) \quad \text{and} \quad \tilde{R}\big(a *_\theta b\big) = -\tilde{R}(a)\tilde{R}(b). \tag{3.7}$$

The result in Proposition 3.2 is best understood in the dendriform setting which we introduce now. A *dendriform algebra* [30] over a field k is a k-vector space A endowed with two bilinear operations $\prec$ and $\succ$ subject to the three axioms below:

$$\begin{aligned}(a \prec b) \prec c &= a \prec (b * c),\\ (a \succ b) \prec c &= a \succ (b \prec c),\\ a \succ (b \succ c) &= (a * b) \succ c,\end{aligned}$$

where $a * b$ stands for $a \prec b + a \succ b$. These axioms easily yield associativity for the law $*$. The bilinear operations $\rhd$ and $\lhd$ defined by

$$a \rhd b := a \succ b - b \prec a, \quad a \lhd b := a \prec b - b \succ a \tag{3.8}$$

are left pre-Lie and right pre-Lie, respectively, which means that we have

$$(a \rhd b) \rhd c - a \rhd (b \rhd c) = (b \rhd a) \rhd c - b \rhd (a \rhd c), \tag{3.9}$$

$$(a \lhd b) \lhd c - a \lhd (b \lhd c) = (a \lhd c) \lhd b - a \lhd (c \lhd b). \tag{3.10}$$

The associative operation $*$ and the pre-Lie operations $\rhd$, $\lhd$ all define the same Lie bracket:

$$[a, b] := a * b - b * a = a \rhd b - b \rhd a = a \lhd b - b \lhd a. \tag{3.11}$$

Proposition 3.3 ([11]). *Any Rota–Baxter algebra gives rise to two dendriform algebra structures given by*

$$a \prec b := aR(b) + \theta ab = -a\tilde{R}(b), \quad a \succ b := R(a)b, \tag{3.12}$$

$$a \prec' b := aR(b), \qquad a \succ' b := R(a)b + \theta ab = -\tilde{R}(a)b. \tag{3.13}$$

The associated associative product $*$ is given for both structures by $a * b = aR(b) + R(a)b + \theta ab$ and thus coincides with the double Rota–Baxter product (3.6).

Remark 3.4 ([11]). In fact, by splitting again the binary operation $\prec$ (or alternatively $\succ'$), any Rota–Baxter algebra is tri-dendriform [32], in the sense that the Rota–Baxter structure yields three binary operations $<, \diamond$ and $>$ subject to axioms refining the axioms of dendriform algebras. The three binary operations are defined by $a < b = aR(b)$, $a \diamond b = \theta ab$ and $a > b = R(a)b$. Choosing to put the operation $\diamond$ to the $<$ or $>$ side gives rise to the two dendriform structures above.

Let $\bar{A} = A \oplus k.\mathbf{1}$ be our dendriform algebra augmented by a unit $\mathbf{1}$:

$$a \prec \mathbf{1} := a =: \mathbf{1} \succ a, \quad \mathbf{1} \prec a := 0 =: a \succ \mathbf{1}, \tag{3.14}$$

implying $a * \mathbf{1} = \mathbf{1} * a = a$. Note that $\mathbf{1} * \mathbf{1} = \mathbf{1}$, but that $\mathbf{1} \prec \mathbf{1}$ and $\mathbf{1} \succ \mathbf{1}$ are not defined [41], [6]. We recursively define the following set of elements of $\bar{A}[\![t]\!]$ for a fixed $x \in A$:

$$\begin{aligned} w_\prec^{(0)}(x) &= w_\succ^{(0)}(x) = \mathbf{1}, \\ w_\prec^{(n)}(x) &:= x \prec \big(w_\prec^{(n-1)}(x)\big), \\ w_\succ^{(n)}(x) &:= \big(w_\succ^{(n-1)}(x)\big) \succ x. \end{aligned}$$

We also define the following set of iterated left and right pre-Lie products (3.8). For $n > 0$, let $a_1, \ldots, a_n \in A$:

$$\ell^{(n)}(a_1, \ldots, a_n) := \big(\ldots\big((a_1 \rhd a_2) \rhd a_3\big) \cdots \rhd a_{n-1}\big) \rhd a_n \tag{3.15}$$

$$r^{(n)}(a_1, \ldots, a_n) := a_1 \lhd \big(a_2 \lhd \big(a_3 \lhd \cdots (a_{n-1} \lhd a_n)\big) \ldots\big). \tag{3.16}$$

For a fixed single element $a \in A$ we can write more compactly for $n > 0$:

$$\ell^{(n+1)}(a) = \big(\ell^{(n)}(a)\big) \rhd a \quad \text{and} \quad r^{(n+1)}(a) = a \lhd \big(r^{(n)}(a)\big), \tag{3.17}$$

and $\ell^{(1)}(a) := a =: r^{(1)}(a)$. We have the following theorem [23], [17].

Theorem 3.5. *We have*

$$w_\succ^{(n)}(a) = \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k>0}} \frac{\ell^{(i_1)}(a) * \cdots * \ell^{(i_k)}(a)}{i_1(i_1+i_2)\ldots(i_1+\cdots+i_k)}$$

and

$$w_\prec^{(n)}(a) = \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k>0}} \frac{r^{(i_k)}(a) * \cdots * r^{(i_1)}(a)}{i_1(i_1+i_2)\ldots(i_1+\cdots+i_k)}.$$

These identities nicely show how the dendriform pre-Lie and associative products fit together. This will become even more evident in the following.

We are interested in the solutions X and Y in $\bar{A}[\![t]\!]$ of the following two equations:

$$X = \mathbf{1} + ta \prec X, \quad Y = \mathbf{1} - Y \succ ta. \tag{3.18}$$

Formal solutions to (3.18) are given by

$$X = \sum_{n\geq 0} t^n w_{\prec}^{(n)}(a) \quad \text{resp.} \quad Y = \sum_{n\geq 0} (-t)^n w_{\succ}^{(n)}(a).$$

Let us introduce the following operators in A, where a is any element of A:

$$\begin{aligned} L_{\prec}[a](b) &:= a \prec b, \quad & L_{\succ}[a](b) &:= a \succ b, \\ R_{\prec}[a](b) &:= b \prec a, \quad & R_{\succ}[a](b) &:= b \succ a, \\ L_{\lhd}[a](b) &:= a \lhd b, \quad & L_{\rhd}[a](b) &:= a \rhd b, \\ R_{\lhd}[a](b) &:= b \lhd a, \quad & R_{\rhd}[a](b) &:= b \rhd a. \end{aligned}$$

We have recently obtained the following *pre-Lie Magnus expansion* [19]:

Theorem 3.6. *Let $\Omega' := \Omega'(ta)$, $a \in A$, be the element of $tA[\![t]\!]$ such that $X = \exp^*(\Omega')$ and $Y = \exp^*(-\Omega')$, where X and Y are the solutions of the two equations (3.18), respectively. This element obeys the following recursive equation:*

$$\Omega'(ta) = \frac{R_{\lhd}[\Omega']}{1-\exp(-R_{\lhd}[\Omega'])}(ta) = \sum_{m\geq 0} (-1)^m \frac{B_m}{m!} R_{\lhd}[\Omega']^m(ta), \tag{3.19}$$

or, alternatively,

$$\Omega'(ta) = \frac{L_{\rhd}[\Omega']}{\exp(L_{\rhd}[\Omega'])-1}(ta) = \sum_{m\geq 0} \frac{B_m}{m!} L_{\rhd}[\Omega']^m(ta), \tag{3.20}$$

where the B_l's are the Bernoulli numbers.

Recall that the Bernoulli numbers are defined via the generating series

$$\frac{z}{\exp(z)-1} = \sum_{m\geq 0} \frac{B_m}{m!} z^m = 1 - \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \cdots,$$

and observe that $B_{2m+3} = 0$, $m \geq 0$.

Suppose that the dendriform structure of A comes from a unital Rota–Baxter algebra of weight θ. The unit (which we denote by 1) has nothing to do with the artificially added unit $\mathbf{1}$ of the underlying dendriform algebra. We extend the Rota–Baxter algebra structure to $\bar{A}$ by setting

$$R(\mathbf{1}) := 1, \quad \widetilde{R}(\mathbf{1}) := -1 \quad \text{and} \quad \mathbf{1}.x = x.\mathbf{1} = 0 \text{ for any } x \in \bar{A}.$$

This is consistent with the axioms (3.14) which in particular yield $\mathbf{1} \succ x = R(\mathbf{1})x$ and $x \prec \mathbf{1} = -x\widetilde{R}(\mathbf{1})$, in coherence with the dendriform axioms. Using (3.12) the pre-Lie products (3.8) write as follows:

$$a \rhd_\theta b = a \succ b - b \prec a = R(a)b + b\widetilde{R}(a) = [R(a), b] - \theta ba \tag{3.21}$$

$$a \lhd_\theta b = a \prec b - b \succ a = -a\widetilde{R}(b) - R(b)a = [a, R(b)] + \theta ab. \tag{3.22}$$

Introduce the *weight $\theta \in k$ pre-Lie Magnus type recursion*, $\Omega'_\theta := \Omega'_\theta(ta) \in tA[[t]]$, where the Rota–Baxter operator R is naturally extended to $\bar{A}[[t]]$ by $k[[t]]$-linearity:

$$\Omega'_\theta(ta) = \sum_{m\geq 0} \frac{B_m}{m!} L_{\rhd_\theta}[\Omega'_\theta]^m(ta). \tag{3.23}$$

First, we observe that the weight zero case, $\theta = 0$, is fully coherent with the origin of Magnus' work [33].

Corollary 3.7. *The limit $\theta \to 0$ of the pre-Lie Magnus type recursion (3.23) reduces to the classical Magnus expansion:*

$$\Omega'_0(ta) = \sum_{m\geq 0} \frac{B_m}{m!} \mathrm{ad}^{(m)}_{R(\Omega'_0)}(ta). \tag{3.24}$$

Here, as usual, $\mathrm{ad}_f(g) := fg - gf := [f, g]$. This follows immediately from (3.21), i.e., $a \rhd_0 b = [R(a), b]$. Recall that Magnus in [33] considers solutions of the classical initial value problem

$$\frac{d}{ds}\Phi(s) = \Psi(s)\Phi(s), \quad \Phi(0) = 1, \tag{3.25}$$

in a non-commutative context, i.e., Ψ and Φ are supposed to be linear operators depending on a real variable t. Here, 1 denotes the identity operator. Magnus obtained a differential equation for the linear operator $\Omega(\Psi)(s)$ depending on $\Psi(s)$, and with $\Omega(\Psi)(0) = 0$, such that

$$Y(s) = \exp\big(\Omega(\Psi)(s)\big) = \exp\left(\int_0^s \dot{\Omega}(\Psi)(u)\,du\right) = \sum_{n\geq 0} \frac{\big(\Omega(\Psi)(s)\big)^n}{n!},$$

leading to the recursively defined classical Magnus expansion:

$$\Omega(t\Psi)(s) = t\int_0^s \Psi(u)\,du + \int_0^s \sum_{n>0} \frac{B_n}{n!} \mathrm{ad}^{(n)}_{\Omega(t\Psi)(u)}(t\Psi(u))\,du, \tag{3.26}$$

where we introduced the parameter t for later use. The expansion (3.24) coincides with (3.26) if the underlying weight zero Rota–Baxter algebra is the one mentioned above, that is, the ring of real continuous functions $C(\mathbb{R})$ with pointwise product and the indefinite Riemann integral as weight zero Rota–Baxter map. Sometimes, Magnus' expansion (3.26) is also called *continuous Baker–Campbell–Hausdorff formula*, e.g. see [25], [38], [47].

Using (3.7), Theorem 3.6 implies for a fixed element $a \in A$ the following corollary which can be interpreted as the non-commutative Spitzer identity.

Corollary 3.8. *Let (A, R) be a Rota–Baxter algebra of weight θ. The elements $\hat{X} := -\tilde{R}(X) = \exp(-\tilde{R}(\Omega'_\theta(ta)))$ and $\hat{Y} := R(Y) = \exp(-R(\Omega'_\theta(ta)))$ in $A[[t]]$*

solve the equations

$$\hat{X} = 1 - t\widetilde{R}(a\hat{X}) \quad \textit{resp.} \quad \hat{Y} = 1 - tR(\hat{Y}a). \tag{3.27}$$

Moreover, these recursions lead to the following theorem due to Atkinson [1].

Theorem 3.9. *Let (A, R) be a Rota–Baxter algebra of weight θ. The recursions (3.27) have the factorization property*

$$\hat{Y}(1 - ta\theta)\hat{X} = 1 \quad \textit{or} \quad (1 - ta\theta) = \hat{Y}^{-1}\hat{X}^{-1}.$$

Proof. Here, 1 is the algebra unit in A. The proof of this statement reduces to a simple algebraic exercise. Recall that $\widetilde{R} = -\theta\,\mathrm{id}_A - R$; then

$$\begin{aligned}
\hat{Y}\hat{X} &= \big(1 - tR(\hat{Y}a)\big)\big(1 - t\widetilde{R}(a\hat{X})\big)\\
&= 1 - t\widetilde{R}(a\hat{X}) - tR(\hat{Y}a) + t^2R(\hat{Y}a)\widetilde{R}(a\hat{X})\\
&= 1 - t\widetilde{R}\big((1 - tR(\hat{Y}a))a\hat{X}\big) - tR\big(\hat{Y}a(1 - tR(a\hat{X}))\big) = 1 + t\theta\hat{Y}a\hat{X}.
\end{aligned}$$

Uniqueness of the factorization follows when R is idempotent. □

Without problems the reader verifies the next corollary.

Corollary 3.10. *The elements $\hat{X}^{-1} = \exp(\widetilde{R}(\Omega'_\theta(ta)))$ and $\hat{Y}^{-1} := R(Y) = \exp(R(\Omega'_\theta(ta)))$ in $A[\![t]\!]$ solve the equations*

$$\hat{X}^{-1} = 1 + t\widetilde{R}(\hat{Y}a) \quad \textit{resp.} \quad \hat{Y}^{-1} = 1 + tR(a\hat{X}). \tag{3.28}$$

Hence, defining the application $\overline{B}(a) := \hat{Y}a$ we may state the two key equations:

$$\hat{Y} = 1 - tR(\overline{B}(a)) \quad \text{and} \quad \hat{X}^{-1} = 1 + t\widetilde{R}(\overline{B}(a)).$$

Corollary 3.8 immediately results in the following lemma.

Lemma 3.11.

$$\overline{B}(ta) = \hat{Y}ta = \exp^{*_\theta}(\Omega'_\theta(ta)) - 1. \tag{3.29}$$

Proof. Indeed, using simple algebra we see that

$$\begin{aligned}
\exp^{*_\theta}(\Omega'_\theta(ta)) - 1 &= \sum_{n=1}^{\infty} \frac{\big(\Omega'_\theta(ta)\big)^{*_\theta n}}{n!}\\
&= \sum_{n=1}^{\infty} \frac{-\theta^{-1}}{n!}\Big(\big(R(\Omega'_\theta(ta))\big)^n - (-1)^n\big(\widetilde{R}(\Omega'_\theta(ta))\big)^n\Big)\\
&= -\sum_{n=0}^{\infty} \frac{\theta^{-1}}{n!}\big(R(\Omega'_\theta(ta))\big)^n + \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{-1}}{n!}\big(\widetilde{R}(\Omega'_\theta(ta))\big)^n\\
&= -\exp\big(R(\Omega'_\theta(ta))\big) + \exp\big(-\widetilde{R}(\Omega'_\theta(ta))\big)
\end{aligned}$$

$$= \exp\big(R(\Omega'_\theta(ta))\big)\big(-1 + \exp\big(-R(\Omega'_\theta(ta))\big)\exp\big(-\tilde{R}(\Omega'_\theta(ta))\big)\big)$$
$$= \exp\big(R(\Omega'_\theta(ta))\big)ta = t\hat{Y}a. \qquad \square$$

Let us take a closer look at Atkinson's theorem. In the light of Corollary 3.8 we find by using the exponential solutions $\hat{X} := -\tilde{R}(X) = \exp(-\tilde{R}(\Omega_\theta(ta)))$ and $\hat{Y} := R(Y) = \exp(-R(\Omega_\theta(ta)))$ in $A[\![t]\!]$ to the recursions (3.27) that

$$1 - \theta ta = \exp(-\theta\alpha_\theta) = \exp\big(R\big(\Omega'_\theta(ta)\big)\big)\exp\big(\tilde{R}\big(\Omega'_\theta(ta)\big)\big) \tag{3.30}$$

with $\alpha_\theta := \alpha_\theta(ta) := -\frac{1}{\theta}\log(1 - \theta ta)$.

When $\theta = 0$ we see immediately that $R = -\tilde{R}$ and the above factorization of the algebra unit becomes very evident. Instead, let us keep $\theta \neq 0$, but assume the underlying Rota–Baxter algebra to be commutative. Recall that $a \rhd_\theta b = -\theta ba$. Hence, we see that

$$\Omega'_\theta(ta) = \sum_{m\geq 0} \frac{B_m}{m!}\big(-\theta\Omega'_\theta\big)^m ta,$$

which, using the generating series for the Bernoulli numbers, is solved by $\Omega'_\theta(ta) = -\theta^{-1}\log(1 - \theta ta)$. Hence, in the commutative setting we find

$$1 - \theta ta = \exp\big(R\big(-\theta^{-1}\log(1 - \theta ta)\big)\big)\exp\big(\tilde{R}\big(-\theta^{-1}\log(1 - \theta ta)\big)\big),$$

which is in full accordance with the classical result due to Spitzer [45]. In fact, Baxter [2] showed for commutative Rota–Baxter algebras (A, R) of weight θ the identity

$$\exp\Big(\sum_{n>0} \frac{r_n t^n}{n}\Big) = 1 + \sum_{m>0} a_m t^m$$

in $A[\![t]\!]$, where $r_n := R(\theta^{n-1}a^n)$, $a_1 = r_1 = R(a)$, and

$$a_m = \sum_{(\lambda_1,\ldots,\lambda_m)} \frac{r_1^{\lambda_1}\ldots r_m^{\lambda_m}}{1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m}\lambda_1!\ldots\lambda_m!} = \underbrace{R\big(R(R(\ldots R}_{m \text{ times}}(a)a)\ldots a)a\big).$$

The sum goes over all integer m-tuples $(\lambda_1, \ldots, \lambda_m)$, $\lambda_i \geq 0$ for which $1\lambda_1 + \cdots + m\lambda_m = m$. The non-commutative generalization of this formulation of Spitzer's identity follows from Theorem 3.5 together with Proposition 3.3. Indeed, with $\Omega'(ta) = \sum_{n>0} t^n \Omega'_{(n)}(a)$ we find

$$\exp\Big(-\sum_{n>0} R(\Omega'_{(n)})t^n\Big) = 1 + \sum_{m>0} R(w_{\succ}^{(m)}(a))(-t)^m$$
$$\exp\Big(-\sum_{n>0} \tilde{R}(\Omega'_{(n)})t^n\Big) = 1 + \sum_{m>0} \tilde{R}(w_{\prec}^{(m)}(a))(-t)^m,$$

corresponding to the recursions

$$\hat{X} = 1 - R(\hat{X}a) \quad \text{resp.} \quad \hat{Y} = 1 - \tilde{R}(a\hat{Y}).$$

Again, but more concretely, in the non-commutative case we see how both, the double Rota–Baxter product (3.6) and the pre-Lie product (3.21) fit together in the algebraic structure of weight θ Rota–Baxter algebras.

3.1 Weight θ BCH-recursion vs. pre-Lie Magnus expansion

In the light of Corollary 2.12 and Atkinson's Theorem, it seems to be natural to compare the weight θ pre-Lie Magnus expansion with the Baker–Campbell–Hausdorff recursion (2.16).

For this, we first introduce the weight θ Baker–Campbell–Hausdorff recursion following simply from a linear filtration preserving map P, such that $-\theta\,\mathrm{id}_A = P + (-\theta\,\mathrm{id}_A - P)$, $\theta \in k$. Hence

$$\chi_\theta(a) = a - \frac{1}{\theta}\,\mathrm{BCH}\big(-P(\chi_\theta(a)),\ -\theta a\big), \tag{3.31}$$

giving rise to the factorization

$$\exp(-\theta a) = \exp\big(P\left(\chi_\theta(a)\right)\big)\exp\big(\tilde{P}\left(\chi_\theta(a)\right)\big). \tag{3.32}$$

Now, let us assume that P is a filtration preserving Rota–Baxter map. Then Corollary 2.12 and Theorem 3.9, respectively equation (3.30) imply the equality

$$\Omega'_\theta(ta) = \chi_\theta(\alpha_\theta) = \chi_\theta\Big(-\frac{\log(1-\theta ta)}{\theta}\Big), \tag{3.33}$$

From (3.33) we get for any $\alpha \in tA[\![\lambda]\!]$,

$$\chi_\theta\big(\alpha_\theta\big) = \Omega'_\theta\Big(\frac{1-\exp(-\theta\alpha)}{\theta}\Big). \tag{3.34}$$

3.2 Application to perturbative renormalization II

We return to paragraph 2.7 where we analysed Connes–Kreimer's factorization from the point of view of the Baker–Campbell–Hausdorff recursion (2.16). Recall that the projector π (respectively its lift to $\mathcal{L} = \mathcal{L}(\mathcal{H}, \mathcal{A})$, denoted by P) is a weight minus one Rota–Baxter map. Hence the Birkhoff–Connes–Kreimer factorization naturally fits into the context of Rota–Baxter algebra, in particular with respect to Atkinson's factorization theorem respectively the non-commutative Spitzer identity. Hence, it follows ultimately that the group $G_1(\mathcal{A})$ decomposes as a set into the product of two subgroups:

$$G_1(\mathcal{A}) = G_1^-(\mathcal{A}) * G_1^+(\mathcal{A}),$$

where

$$G_1^-(\mathcal{A}) = \exp^*\big(P(\mathcal{A})\big) \quad \text{and} \quad G_1^+(\mathcal{A}) = \exp^*\big(\tilde{P}(\mathcal{A})\big).$$

Now, using (3.34) we see that the counterterm character φ_- in the decomposition

$$\varphi = \exp^*(a) = \varphi_-^{-1} * \varphi_+$$

writes as

$$\varphi_- = \exp^*\big(-P\big(\Omega'_{-1}(\exp^*(a) - e)\big)\big),$$

where $\exp^*(a) - e = \varphi - e$ already appeared in the context of (2.8) respectively (2.9), see also Proposition 2.7. Using Lemma 3.11, Bogoliubov's preparation map $B(\varphi) := \varphi_- * (\varphi - e)$ finds its exponential form

$$B(\varphi) = \exp^{*_{-1}}\big(\Omega'_{-1}(\varphi - e)\big) - e,$$

such that the non-commutative Spitzer identity implies Bogoliubov's recursion for φ_-. Moreover, Corollary 3.10 tells immediately the equation for φ_+. Here, $*_{-1}$ stands for the Rota–Baxter double product in the weight minus one Rota–Baxter algebra $(\mathcal{L}(\mathcal{H}, \mathcal{A}), P)$.

Recall our characterization of the Baker–Campbell–Hausdorff recursion (2.16) as the analog of Bogoliubov's preparation map on the Lie algebra $\mathfrak{g}_1(\mathcal{A})$ of infinitesimal characters. Now, by recalling Proposition 2.7 we may identify the weight minus one pre-Lie Magnus expansion Ω'_{-1} as the analog of Bogoliubov's preparation map. More precisely, first remember that $\mathfrak{g}(\mathcal{A})$ contains $\mathfrak{g}_1(\mathcal{A})$ as a Lie subalgebra. Hence, let $\varphi \in G_1(\mathcal{A}) \subset G(\mathcal{A})$, i.e., $\varphi = e + (\varphi - e)$, where obviously $a := \varphi - e \in \mathfrak{g}(\mathcal{A})$. Then

$$\Omega'_{-1}(a) = \sum_{m \geq 0} \frac{B_m}{m!} L_{\rhd_{-1}}[\Omega'_{-1}]^m(a) \tag{3.35}$$

maps $a \in \mathfrak{g}(\mathcal{A}) \to \mathfrak{g}_1(\mathcal{A})$, such that $B(\varphi) = \exp^{*_{-1}}\big(\Omega'_{-1}(\varphi - e)\big) - e$ and

$$\varphi_- = \exp^*\big(-P\big(\Omega'_{-1}(\varphi - e)\big)\big) \quad \text{and} \quad \varphi_+ = \exp^*\big(\tilde{P}\big(\Omega'_{-1}(\varphi - e)\big)\big)$$

solve

$$\varphi_- = e - P\big(\varphi_- * (\varphi - e)\big) \quad \text{and} \quad \varphi_+ = e + \tilde{P}\big(\varphi_- * (\varphi - e)\big),$$

respectively.

We remark here that this purely Lie algebraic approach to renormalization is an extension of earlier work [17] and will be further explored in the near future. Let us mention that the results in [17] rely on both, the properties of the Dynkin idempotent and on properties of Hopf algebras encapsulated in the notion of associated descent algebras. Similarly, in [22], see also [18], we use free Lie algebra theory, i.e., Lie idempotents to achieve a closed form for the Bogoliubov recursion.

3.3 Non-commutative Bohnenblust–Spitzer formulas

Let n be a positive integer, and let $\mathcal{OP}_n$ be the set of ordered partitions of $\{1, \ldots, n\}$, i.e., sequences $(\pi_1, \ldots, \pi_k)$ of disjoint subsets (*blocks*) whose union is $\{1, \ldots, n\}$.

We denote by $\mathcal{OP}_n^k$ the set of ordered partitions of $\{1,\dots,n\}$ with k blocks. Let us introduce for any $\pi \in \mathcal{OP}_n^k$ the coefficient

$$\omega(\pi) = \frac{1}{|\pi_1|(|\pi_1|+|\pi_2|)\dots(|\pi_1|+|\pi_2|+\dots+|\pi_k|)}.$$

Theorem 3.12. *Let $a_1,\dots,a_n$ be elements in a dendriform algebra A. For any subset $E=\{j_1,\dots,j_m\}$ of $\{1,\dots,n\}$ let $\ell(E)\in A$ be defined by*

$$\ell(E) := \sum_{\sigma\in S_m} \ell^{(m)}(a_{j_{\sigma_1}},\dots,a_{j_{\sigma_m}}).$$

We have

$$\sum_{\sigma\in S_n} \big(\dots(a_{\sigma_1}\succ a_{\sigma_2})\succ\cdots a_{\sigma_{n-1}}\big)\succ a_{\sigma_n} = \sum_{k\geq 1}\sum_{\pi\in\mathcal{OP}_n^k} \omega(\pi)\ell(\pi_1)*\dots*\ell(\pi_k).$$

See [22] where this identity is settled in the Rota–Baxter setting, see also [18]. The proof in the dendriform context is entirely similar. Another expression for the left-hand side can be obtained [23]: For any permutation $\sigma\in S_n$ we define the element $T_\sigma(a_1,\dots,a_n)$ as follows: define first the subset $E_\sigma\subset\{1,\dots,n\}$ by $k\in E_\sigma$ if and only if $\sigma_{k+1}>\sigma_j$ for any $j\leq k$. We write E_σ in the increasing order:

$$1\leq k_1<\dots<k_p\leq n-1.$$

Then we set

$$T_\sigma(a_1,\dots,a_n) := \ell^{(k_1)}(a_{\sigma_1},\dots,a_{\sigma_{k_1}})*\dots*\ell^{(n-k_p)}(a_{\sigma_{k_p+1}},\dots,a_{\sigma_n}). \tag{3.36}$$

There are $p+1$ packets separated by p stars in the right-hand side of the expression (3.36) above, and the parentheses are set to the left inside each packet. Following [29] it is convenient to write a permutation by putting a vertical bar after each element of E_σ. For example for the permutation $\sigma=(3261457)$ inside S_7 we have $E_\sigma=\{2,6\}$. Putting the vertical bars

$$\sigma = (32|6145|7),$$

we see that the corresponding element in A will then be

$$\begin{aligned} T_\sigma(a_1,\dots,a_7) &= \ell^{(2)}(a_3,a_2)*\ell^{(4)}(a_6,a_1,a_4,a_5)*\ell^{(1)}(a_7)\\ &= (a_3\rhd a_2)*\Big(\big((a_6\rhd a_1)\rhd a_4\big)\rhd a_5\Big)*a_7. \end{aligned}$$

Theorem 3.13. *For any $a_1,\dots,a_n$ in the dendriform algebra A the following identity holds:*

$$\sum_{\sigma\in S_n} (\dots(a_{\sigma_1}\succ a_{\sigma_2})\succ\cdots)\succ a_{\sigma_n} = \sum_{\sigma\in S_n} T_\sigma(a_1,\dots,a_n). \tag{3.37}$$

A q-analog of this identity has been recently proved by J.-C. Novelli and J.-Y. Thibon [39].

4 As simple as it gets: a matrix calculus for renormalization

We shortly introduce a matrix setting for renormalization, associated with any left co-ideal of the Hopf algebra. Although we will not detail this point, let us mention that this matrix approach is particularly well-suited for the study of the renormalization group and the beta-function for local characters in connected *graded* Hopf algebras with values into meromorphic functions [9]. See [21] as well as [20], [14] for a detailed account and applications.

4.1 The matrix representation

In this section we introduce the matrix representation of $\mathcal{L}(\mathcal{H},\mathcal{A})$ associated with a left coideal, following [21]. Let $\mathcal{H}$ be a connected filtered Hopf algebra over the field k, let $\mathcal{A}$ be any commutative unital k-algebra, and let $\big(\mathcal{L}(\mathcal{H},\mathcal{A}),\star\big)$ be the algebra of k-linear maps from $\mathcal{H}$ to $\mathcal{A}$ endowed with the convolution product. Let J be any left coideal of $\mathcal{H}$ (i.e., a vector subspace of $\mathcal{H}$ such that $\Delta(J)\subset\mathcal{H}\otimes J$).

We fix a basis $X=(x_i)_{i\in I}$ of the left coideal J. Furthermore we suppose that this basis is denumerable (hence indexed by $I=\mathbb{N}$ or $I=\{1,\ldots,m\}$) and *filtration ordered*, i.e., such that if $i\le j$ and $x_j\in\mathcal{H}^n$, then $x_i\in\mathcal{H}^n$.

Definition 4.1. The *coproduct matrix* in the basis X is the $|I|\times|I|$ matrix M with entries in $\mathcal{H}$ defined by

$$\Delta(x_i)=\sum_{j\in I}M_{ij}\otimes x_j.$$

The coproduct matrix is lower-triangular with diagonal terms equal to $\mathbf{1}$ ([21] Lemma 1). Now define $\Psi_J:\mathcal{L}(\mathcal{H},\mathcal{A})\to\mathrm{End}_{\mathcal{A}}(\mathcal{A}\otimes J)$ by

$$\Psi_J[f](x_j)=\sum_i f(M_{ij})\otimes x_i. \tag{4.1}$$

In other words, the matrix of $\Psi_J[f]$ is given by $f(M):=\big(f(M_{ij})\big)_{i,j\in I}$. It is shown in [14] and also [21] that the map Ψ_J defined above is an algebra homomorphism. Its transpose does not depend on the choice of the basis. The Lie algebra of $\mathcal{A}$-valued infinitesimal characters (resp. the group of $\mathcal{A}$-valued characters) is mapped by Ψ_J into the Lie subalgebra of strictly lower-triangular matrices (resp. into the group of lower triangular matrices with – $\mathcal{A}$-algebra units – 1's on the diagonal).

The coproduct matrix M with entries in $\mathcal{H}$ can be seen as the image of the identity map under $\Psi_J:\mathcal{L}(\mathcal{H},\mathcal{H})\to\mathrm{End}_{\mathcal{H}}(\mathcal{H}\otimes J)$, i.e.

$$\Psi_J[\mathrm{Id}](x_j)=\sum_i \mathrm{Id}(M_{ij})\otimes x_i. \tag{4.2}$$

We have $\Psi_J[S] = M^{-1}$, where S is the antipode. The matrix $L = \log M$ is the matrix of *normal coordinates*. For any $\mathcal{A}$-valued character φ we have

$$\log \Psi_J[\varphi] = \varphi(L).$$

4.2 The matrix form of Connes–Kreimer's Birkhoff decomposition

Suppose that the commutative target space algebra $\mathcal{A}$ in $\mathcal{L}(\mathcal{H}, \mathcal{A})$ splits into two subalgebras

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+, \tag{4.3}$$

where the unit $1_{\mathcal{A}}$ belongs to $\mathcal{A}_+$. Let us denote by $\pi : \mathcal{A} \to \mathcal{A}_-$ the projection onto $\mathcal{A}_-$ parallel to $\mathcal{A}_+$, which is a weight -1 Rota–Baxter map. The algebra $\mathcal{M} := \mathcal{M}^{\ell}_{|I|}(\mathcal{A})$ of lower-triangular $|I| \times |I|$-matrices with coefficients in $\mathcal{A}$ is filtered by the subalgebras $\mathcal{M}^i = \{X \in \mathcal{M},\ X_{kl} = 0 \text{ if } k < l + i\}$. The filtration is finite, hence complete.

We define a Rota–Baxter map R on $\Psi_J[\mathcal{L}(\mathcal{H}, \mathcal{A})] \subset \mathcal{M}^{\ell}_{|I|}(\mathcal{A})$ by extending the Rota–Baxter map π on $\mathcal{A}$ entrywise, i.e., for the matrix $\tau = (\tau_{ij}) \in \mathcal{M}^{\ell}_{|I|}(\mathcal{A})$, define

$$\mathrm{R}(\tau) = \big(\pi(\tau_{ij})\big). \tag{4.4}$$

The algebra $\mathcal{M}$ is then a complete filtered Rota–Baxter algebra. Let us denote $\Psi_J[\varphi] := \widehat{\varphi}$ for short. As $\varphi \mapsto \widehat{\varphi}$ is a morphism of complete filtered Rota–Baxter algebras we immediately get the matrix Birkhoff–Connes–Kreimer decomposition:

$$\widehat{\varphi} = \widehat{\varphi_-}^{-1}\widehat{\varphi_+}.$$

In other words, the map Ψ_J respects the Birkhoff decomposition, i.e., $\widehat{\varphi}_{\pm} = \widehat{\varphi_{\pm}}$. We immediately see that $\widehat{\varphi}_-$ and $\widehat{\varphi}_+^{-1}$ are unique solutions of the following equations:

$$\widehat{\varphi}_- = \mathbf{1} - \mathrm{R}\big(\widehat{\varphi}_-\, (\widehat{\varphi} - \mathbf{1})\big), \tag{4.5}$$

$$\widehat{\varphi}_+^{-1} = \mathbf{1} - \widetilde{\mathrm{R}}\big((\widehat{\varphi} - \mathbf{1})\, \widehat{\varphi}_+^{-1}\big), \tag{4.6}$$

respectively. Moreover, after some simple algebra using the matrix factorization $\widehat{\varphi} = \widehat{\varphi}_-^{-1}\widehat{\varphi}_+$

$$\widehat{\varphi}_+(\widehat{\varphi}^{-1} - \mathbf{1}) \; = \widehat{\varphi}_- - \widehat{\varphi}_+ = -\widehat{\varphi}_-(\widehat{\varphi} - \mathbf{1})$$

we immediately get the recursion for $\widehat{\varphi}_+$ [21]

$$\widehat{\varphi}_+ = \mathbf{1} - \widetilde{\mathrm{R}}\big(\widehat{\varphi}_+\, (\widehat{\varphi}^{-1} - \mathbf{1})\big), \tag{4.7}$$

and hence we see that

$$\widehat{\varphi}_+ = \mathbf{1} + \widetilde{\mathrm{R}}\big(\widehat{\varphi}_-\, (\widehat{\varphi} - \mathbf{1})\big). \tag{4.8}$$

The matrix entries of $\widehat{\varphi}_-$ and $\widehat{\varphi}_+^{-1}$ can be calculated without recursions using $\sigma := \widehat{\varphi}$ from the equations [20]:

$$(\widehat{\varphi}_-)_{ij} = -\pi(\sigma_{ij}) - \sum_{k=2}^{j-i} \sum_{i>l_1>\cdots>l_{k-1}>j} (-1)^{k+1}\pi\big(\pi(\ldots\pi(\sigma_{il_1})\sigma_{l_1l_2})\ldots\sigma_{l_{k-1}j}\big)$$

$$\begin{aligned}(\widehat{\varphi}_+^{-1})_{ij} &= -\tilde{\pi}((\sigma^{-1})_{ij}) \\ &\quad - \sum_{k=2}^{j-i} \sum_{i>l_1>\cdots>l_{k-1}>j} (-1)^{k+1}\tilde{\pi}\big(\tilde{\pi}(\ldots\tilde{\pi}((\sigma^{-1})_{il_1})(\sigma^{-1})_{l_1l_2})\ldots(\sigma^{-1})_{l_{k-1}j}\big),\end{aligned}$$

where $\tilde{\pi} := \mathrm{id}_{\mathcal{A}} - \pi$. The matrix entries of $\widehat{\varphi}_+$ follow from the first formula, i.e., the one for the entries in $\widehat{\varphi}_-$, by replacing π by $-\tilde{\pi}$. We may therefore define the matrix

$$\widehat{B}[\varphi] := \widehat{\varphi}_- \,(\widehat{\varphi} - \mathbf{1}) \tag{4.9}$$

such that

$$\widehat{\varphi}_- = \mathbf{1} - \mathrm{R}\big(\widehat{B}[\varphi]\big) \quad \text{and} \quad \widehat{\varphi}_+ = \mathbf{1} + \widetilde{\mathrm{R}}\big(\widehat{B}[\varphi]\big). \tag{4.10}$$

In fact, equations (4.8) and (4.6) may be called *Bogoliubov's matrix formulae* for the counter term and renormalized Feynman rules matrix, $\widehat{\varphi}_-$, $\widehat{\varphi}_+$, respectively. Equation (4.9) is the matrix form of Bogoliubov's preparation map (2.6), e.g. see [6]:

$$\widehat{B}[\varphi] := \Psi_J[B(\varphi)] = \Psi_J[\varphi_- \star (\varphi - e)]. \tag{4.11}$$

Remark 4.2. We may apply the result from Subsection 2.6 to the above matrix representation of $\mathfrak{g}_{\mathcal{A}}$ respectively $G_{\mathcal{A}}$. We have shown the existence of a unique non-linear map χ on $\mathfrak{g}_{\mathcal{A}}$ which allows to write the characters φ_- and φ_+ as exponentials. In the matrix picture we hence find for $\widehat{Z} \in \widehat{\mathfrak{g}}_{\mathcal{A}}$ and $\widehat{\varphi} = \exp(\widehat{Z}) \in \widehat{G}_{\mathcal{A}}$:

$$\widehat{\varphi} = \exp\big(\mathrm{R}(\chi(\widehat{Z}))\big)\exp\big(\widetilde{\mathrm{R}}(\chi(\widehat{Z}))\big). \tag{4.12}$$

The matrices $\widehat{\varphi}_- := \exp\big(-\mathrm{R}(\chi(\widehat{Z}))\big)$ and $\widehat{\varphi}_+^{-1} := \exp\big(-\widetilde{\mathrm{R}}(\chi(\widehat{Z}))\big)$ are in $\widehat{G}_{\mathcal{A}}^-$ and $\widehat{G}_{\mathcal{A}}^+$, respectively, and solve Bogoliubov's matrix formulae in (4.10).

Acknowledgments. We would like to thank the organizers of the CIRM 2006 Workshop "Renormalization and Galois Theory" for giving us the opportunity to present parts of our recent research results. The stimulating atmosphere we found at this memorable workshop affected us considerably. Many of the newer results presented here were achieved jointly with J. M. Gracia-Bondía, and F. Patras and a warm thanks goes to both for constant and fruitful ongoing collaborations.

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(Non)commutative Hopf algebras of trees and (quasi)symmetric functions

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Abstract. The Connes–Kreimer Hopf algebra of rooted trees, its dual, and the Foissy Hopf algebra of planar rooted trees are related to each other and to the well-known Hopf algebras of symmetric and quasi-symmetric functions via a pair of commutative diagrams. We show how this point of view can simplify computations in the Connes–Kreimer Hopf algebra and its dual, particularly for combinatorial Dyson–Schwinger equations.

1 Introduction

Hopf algebra techniques were introduced into the study of renormalization in quantum field theory by Connes and Kreimer [6]. The Hopf algebra defined by Connes and Kreimer (in its undecorated form), denoted here by $\mathcal{H}_K$, is the free commutative algebra on the rooted trees, with a coproduct that can be described in terms of "cuts" of rooted trees (see §4 below). The Hopf algebra $\mathcal{H}_K$ is the graded dual of another Hopf algebra (which we call $k\mathcal{T}$), studied earlier by Grossman and Larson [13], whose elements are rooted trees with a noncommutative multiplication.

A noncommutative version of $\mathcal{H}_K$, denoted here by $\mathcal{H}_F$, was introduced by Foissy [7]: unlike $\mathcal{H}_K$, it is self-dual. As shown by Holtkamp [15], $\mathcal{H}_F$ is isomorphic to the Hopf algebra $k[Y_\infty]$ of planar binary trees defined by Loday and Ronco [16]. Foissy [8] showed $\mathcal{H}_F$ isomorphic to the "photon" Hopf algebra $\mathcal{H}^\gamma$ defined by Brouder and Frabetti [3, 4] in connection with renormalization. Here we define a Hopf algebra $k\mathcal{P}$, based on planar rooted trees in the same way $k\mathcal{T}$ is based on rooted trees, which is isomorphic to $\mathcal{H}_F^* \cong \mathcal{H}_F$. Our main purpose is to show how calculations in $\mathcal{H}_K$ and $k\mathcal{T}$ can be simplified by "lifting" them to $\mathcal{H}_F \cong k\mathcal{P}$.

After establishing a result on duality of graded connected Hopf algebras in §2, we briefly introduce in §3 some Hopf algebras familiar in combinatorics: the Hopf algebras Sym of symmetric functions, QSym of quasi-symmetric functions [11], and NSym of noncommutative symmetric functions [10]. Then we discuss, in parallel fashion, the Hopf algebras $k\mathcal{T}$ and $\mathcal{H}_K$ in §4, and $k\mathcal{P}$ and $\mathcal{H}_F$ in §5. In §6 we relate all these Hopf algebras by a pair of commutative diagrams, which we then apply to some calculations. First (in §6.1) we discuss families of elements of $k\mathcal{T}$ that parallel some familiar symmetric functions, and show how symmetric-function identities can be used to obtain results for rooted trees. Then in §6.2 we exhibit explicit solutions of some combinatorial Dyson–Schwinger equations in $\mathcal{H}_K$, and show that these solutions generate sub-Hopf-algebras of $\mathcal{H}_K$. Similar results on Dyson–Schwinger equations were obtained by Bergbauer and Kreimer [1] using different methods.

2 Graded connected Hopf algebras

Let $\mathcal{A}$ be a unital algebra (associative but not necessarily commutative) over a field k of characteristic 0. We assume $\mathcal{A}$ is graded, i.e.,

$$\mathcal{A} = \bigoplus_{n \ge 0} \mathcal{A}_n$$

with $\mathcal{A}_n \mathcal{A}_m \subset \mathcal{A}_{n+m}$. Necessarily $1 \in \mathcal{A}_0$: we shall assume that $\mathcal{A}$ is connected, that is, $\mathcal{A}_0 = k1$.

A coalgebra structure on $\mathcal{A}$ consists of linear functions $\varepsilon\colon \mathcal{A} \to k$ (counit) and $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (coproduct), such that ε sends $1 \in \mathcal{A}_0$ to $1 \in k$ and all positive-degree elements of $\mathcal{A}$ to 0, and Δ respects the grading. These functions must satisfy

$$(\mathrm{id}_{\mathcal{A}} \otimes \varepsilon)\Delta = (\varepsilon \otimes \mathrm{id}_{\mathcal{A}})\Delta = \mathrm{id}_{\mathcal{A}}\,. \tag{2.1}$$

We also assume that Δ is coassociative, in the sense that $\Delta(\Delta \otimes \mathrm{id}_{\mathcal{A}}) = \Delta(\mathrm{id}_{\mathcal{A}} \otimes \Delta)$. For $\mathcal{A}$ to be a Hopf algebra, Δ must be a homomorphism of graded algebras.

Writing the comultiplication applied to $u \in \mathcal{A}$ as

$$\Delta(u) = \sum_u u' \otimes u'', \tag{2.2}$$

we note that condition (2.1) requires that it have the form

$$u \otimes 1 + \sum_{|u'|,|u''|>0} u' \otimes u'' + 1 \otimes u.$$

If $\Delta(u) = u \otimes 1 + 1 \otimes u$, then u is primitive.

A Hopf algebra $\mathcal{A}$ has an antipode $S\colon \mathcal{A} \to \mathcal{A}$, which is an antiautomorphism of $\mathcal{A}$ with the properties that $S(1) = 1$ and

$$\sum_u S(u')u'' = \sum_u u'S(u'') = 0$$

for any u of positive degree, where u', u'' are given by (2.2). Hence $S(u) = -u$ if u is primitive. If $\mathcal{A}$ is either commutative or cocommutative (i.e., $T\Delta = \Delta$, where $T(a \otimes b) = b \otimes a$), then $S^2 = \mathrm{id}_{\mathcal{A}}$.

All the Hopf algebras we consider are locally finite, i.e., $\mathcal{A}_k$ is finite-dimensional for all k. It follows that the (graded) dual $\mathcal{A}^*$ of $\mathcal{A}$ is also a Hopf algebra, with multiplication Δ^* and coproduct μ^* (where $\mu\colon \mathcal{A} \otimes A \to \mathcal{A}$ is the product on $\mathcal{A}$). The Hopf algebra $\mathcal{A}$ is cocommutative if and only if $\mathcal{A}^*$ is commutative.

By an inner product on a graded connected Hopf algebra $\mathcal{A}$, we mean a nondegenerate symmetric linear function $(\cdot,\cdot)\colon \mathcal{A} \otimes \mathcal{A} \to k$ such that $(a,b) = 0$ for homogeneous $a, b \in \mathcal{A}$ of different degrees. The following result gives a criterion to establish when two Hopf algebras $\mathcal{A}$ and $\mathcal{B}$ are dual (i.e., $\mathcal{A}^* \cong \mathcal{B}$).

Theorem 2.1. *Let $\mathcal{A}$, $\mathcal{B}$ be graded connected locally finite Hopf algebras over k which admit inner products $(\cdot,\cdot)_{\mathcal{A}}$ and $(\cdot,\cdot)_{\mathcal{B}}$ respectively. Then $\mathcal{A}$ and $\mathcal{B}$ are dual Hopf algebras provided there is a degree-preserving linear map $\phi\colon \mathcal{A} \to \mathcal{B}$ such that, for all $a_1, a_2, a_3 \in \mathcal{A}$,*

(a) $(a_1, a_2)_{\mathcal{A}} = (\phi(a_1), \phi(a_2))_{\mathcal{B}}$;

(b) $(a_1 a_2, a_3)_{\mathcal{A}} = (\phi(a_1) \otimes \phi(a_2), \Delta(\phi(a_3)))_{\mathcal{B}}$;

(c) $(a_1 \otimes a_2, \Delta(a_3))_{\mathcal{A}} = (\phi(a_1)\phi(a_2), \phi(a_3))_{\mathcal{B}}$.

Proof. Define a linear function $\chi\colon \mathcal{B} \to \mathcal{A}^*$ by $\langle \chi(b), a\rangle = (b, \phi(a))_{\mathcal{B}}$. Injectivity follows from nondegeneracy of the inner products, and since $\mathcal{A}$ and $\mathcal{B}$ are locally finite it follows that χ is a bijection. It remains to show χ a homomorphism, i.e.,

$$\langle \chi(\Delta(b)), a_1 \otimes a_2\rangle = \langle \chi(b), a_1 a_2\rangle \quad \text{for all } b \in \mathcal{B} \text{ and } a_1, a_2 \in \mathcal{A} \tag{2.3}$$

and

$$\langle \chi(b_1 b_2), a\rangle = \langle \chi(b_1) \otimes \chi(b_2), \Delta(a)\rangle \quad \text{for all } b_1, b_2 \in \mathcal{B} \text{ and } a \in \mathcal{A}. \tag{2.4}$$

For (2.3), we have

$$\begin{aligned}
\langle \chi(\Delta(b)), a_1 \otimes a_2\rangle &= (\Delta(b), \phi(a_1) \otimes \phi(a_2))_{\mathcal{B}} \\
&= (\phi^{-1}(b), a_1 a_2)_{\mathcal{A}} \\
&= (b, \phi(a_1 a_2))_{\mathcal{B}} \\
&= \langle \chi(b), a_1 a_2\rangle.
\end{aligned}$$

For (2.4), we have

$$\begin{aligned}
\langle \chi(b_1 b_2), a\rangle &= (b_1 b_2, \phi(a))_{\mathcal{B}} \\
&= (\phi^{-1}(b_1) \otimes \phi^{-1}(b_2), \Delta(a))_{\mathcal{A}} \\
&= \sum_a (\phi^{-1}(b_1), a')_{\mathcal{A}} (\phi^{-1}(b_2), a'')_{\mathcal{A}} \\
&= \sum_a (b_1, \phi(a'))_{\mathcal{B}} (b_2, \phi(a''))_{\mathcal{B}} \\
&= \sum_a \langle \chi(b_1), a'\rangle \langle \chi(b_2), a''\rangle \\
&= \langle \chi(b_1) \otimes \chi(b_2), \Delta(a)\rangle.
\end{aligned}$$

□

We note that it follows from this result that a graded connected locally finite Hopf algebra $\mathcal{A}$ is self-dual provided it admits an inner product $(\cdot,\cdot)$ such that

$$(a_1 \otimes a_2, \Delta(a_3)) = (a_1 a_2, a_3)$$

for all $a_1, a_2, a_3 \in \mathcal{A}$.

3 Symmetric and quasi-symmetric functions

Let $\mathcal{B}$ be the subalgebra of the formal power series ring $k[\![t_1, t_2, \dots]\!]$ consisting of those formal power series of bounded degree, where each t_i has degree 1. An element $f \in \mathcal{B}$ is called a symmetric function if the coefficients in f of the monomials

$$t_{n_1}^{i_1} t_{n_2}^{i_2} \dots t_{n_k}^{i_k} \quad \text{and} \quad t_1^{i_1} t_2^{i_2} \dots t_k^{i_k} \tag{3.1}$$

agree for any sequence of distinct positive integers $n_1, n_2, \dots, n_k$: an element $f \in \mathcal{B}$ is called a quasi-symmetric function if the coefficients in f of the monomials (3.1) agree for any strictly increasing sequence $n_1 < n_2 < \cdots < n_k$ of positive integers. The sets of symmetric and quasi-symmetric functions are denoted Sym and QSym respectively: both are subalgebras of $\mathcal{B}$, and evidently Sym $\subset$ QSym.

As a vector space, QSym is generated by the monomial quasi-symmetric functions M_I, which are indexed by compositions (finite sequences) of positive integers: for $I = (i_1, \dots, i_k)$,

$$M_I = \sum_{n_1 < n_2 < \cdots < n_k} t_{n_1}^{i_1} t_{n_2}^{i_2} \dots t_{n_k}^{i_k}.$$

If we forget order in a composition, we get a partition: a vector-space basis for Sym is given by the monomial symmetric functions

$$m_\lambda = \sum_{\phi(I)=\lambda} M_I,$$

where ϕ is the function from compositions to partitions that forgets order. For example, $m_{2,1,1} = M_{(2,1,1)} + M_{(1,2,1)} + M_{(1,1,2)}$.

It is well known that Sym, as an algebra, is freely generated by several sets of symmetric functions (see, e.g., [17]):

(1) The elementary symmetric functions $e_k = m_{1^k}$ (where 1^k means 1 repeated k times);

(2) The complete symmetric functions

$$h_k = \sum_{|\lambda|=k} m_\lambda = \sum_{|I|=k} M_I;$$

(3) The power-sum symmetric functions $p_k = m_k$.

There is a duality between the e_k and the h_k, reflected in the (graded) identity

$$(1 + e_1 + e_2 + \cdots)(1 - h_1 + h_2 - \cdots) = 1 \tag{3.2}$$

There is also a well-known Hopf algebra structure on Sym [9]. This structure can be defined by making the elementary symmetric functions divided powers, i.e.,

$$\Delta(e_k) = \sum_{i+j=k} e_i \otimes e_j.$$

Equivalently, the h_i are required to be divided powers, or the p_i primitives. For this Hopf algebra structure,

$$\Delta(m_\lambda) = \sum_{\lambda=\mu\cup\nu} m_\mu \otimes m_\nu, \tag{3.3}$$

where the sum is over all pairs (μ, ν) such that $\mu \cup \nu = \lambda$ as multisets. For example, $\Delta(m_{2,1,1})$ is

$$1 \otimes m_{2,1,1} + m_1 \otimes m_{2,1} + m_2 \otimes m_{1,1} + m_{1,1} \otimes m_2 + m_{2,1} \otimes m_1 + m_{2,1,1} \otimes 1.$$

The Hopf algebra Sym is commutative and cocommutative, so its antipode S is an algebra isomorphism with $S^2 = \mathrm{id}$. In fact, as follows from (3.2), $S(e_i) = (-1)^i h_i$. To see that Sym is self-dual, note that it admits an inner product $(\cdot, \cdot)$ such that

$$(h_\lambda, m_\mu) = \delta_{\lambda,\mu}$$

for all partitions λ, μ (where h_λ means $h_{\lambda_1} h_{\lambda_2} \ldots$ for $\lambda = \lambda_1, \lambda_2, \ldots$) [17, §I.4]. Then by equation (3.3),

$$(h_\mu h_\nu, m_\lambda) = (h_\mu \otimes h_\nu, \Delta(m_\lambda)) = \delta_{\mu\cup\nu,\lambda}$$

so Sym is self-dual by Theorem 2.1.

To give QSym the structure of a graded connected Hopf algebra, one defines a coproduct Δ by

$$\Delta(M_{(p_1,\ldots,p_k)}) = \sum_{j=0}^{k} M_{(p_1,\ldots,p_j)} \otimes M_{(p_{j+1},\ldots,p_k)}.$$

This coproduct extends that on Sym, but it is no longer cocommutative: for example,

$$\Delta(M_{(2,1,1)}) = 1 \otimes M_{(2,1,1)} + M_2 \otimes M_{(2,1)} + M_{(2,1)} \otimes M_1 + M_{(2,1,1)} \otimes 1.$$

The antipode of QSym is given by [5, Prop. 3.4]

$$S(M_I) = (-1)^{\ell(I)} \sum_{J \preceq I} M_{\bar{J}},$$

where $\preceq$ is the refinement order on compositions and $\bar{I}$ is the reverse of I.

Since QSym is commutative but not cocommutative, it cannot be self-dual: in fact, its dual is the Hopf algebra NSym of noncommutative symmetric functions in the sense of Gelfand *et al.* [10]. As an algebra NSym is the noncommutative polynomial algebra $k\langle E_1, E_2, \dots \rangle$, with E_i in degree i, and the Hopf algebra structure is determined by declaring the E_i divided powers. There is an abelianization homomorphism $\tau\colon$ NSym $\to$ Sym sending E_i to the elementary symmetric function e_i: its dual $\tau^*\colon$ Sym $\to$ QSym is the inclusion Sym $\subset$ QSym.

4 Hopf algebras of rooted trees

A rooted tree is a partially ordered set with a unique maximal element such that, for any element v, the set of elements exceeding v in the partial order forms a chain. We call the elements of a rooted tree vertices, the maximal element the root, and the minimal elements leaves. If a vertex v covers w in the partial order, we call v the parent of w and w a child of v. We visualize a rooted tree as a directed graph with an edge from each vertex to each of its children: the root (uniquely) has no incoming edges, and leaves have no outgoing edges. Let $\mathcal{T}$ be the set of rooted trees, and

$$\mathcal{T}_n = \{t \in \mathcal{T} : |t| = n+1\}$$

the set of rooted trees with $n+1$ vertices. There is a graded vector space

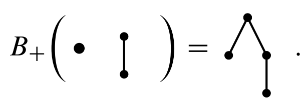
$$k\mathcal{T} = \bigoplus_{n\geq 0} k\mathcal{T}_n$$

with the set of rooted trees as basis.

Each rooted tree t has a symmetry group Symm(t), the group of automorphisms of t as a poset (or directed graph). This group can be explicitly described as follows. For each vertex v of a rooted tree t, let t_v be the rooted tree consisting of v and its descendants (with the partial order inherited from t). If the set of children of v is $C(v) = \{v_1, \dots, v_k\}$, let $SG(t, v)$ be the group of permutations of $C(v)$ generated by those that exchange v_i with v_j when t_{v_i} and t_{v_j} are isomorphic rooted trees. Then

$$\mathrm{Symm}(t) = \prod_{\text{vertices } v \text{ of } t} SG(t, v).$$

By a forest we mean a monomial in rooted trees, with the rooted trees thought of as commuting with each other. There is an algebra of forests, which is just the symmetric algebra $S(k\mathcal{T})$ over $k\mathcal{T}$: the multiplication can be thought of as juxtaposition of forests. For any forest $t_1 t_2 \dots t_k$, there is a rooted tree $B_+(t_1 t_2 \dots t_k)$ given by attaching a new root vertex to each of the roots of $t_1, t_2, \dots, t_k$, e.g.,

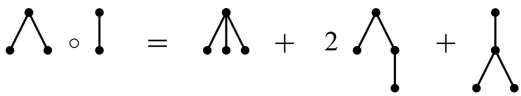


Also, let B_+ send $1 \in S^0(k\mathcal{T})$ (thought of as the empty forest) to $\bullet \in \mathcal{T}_0$. If we grade $S(k\mathcal{T})$ by

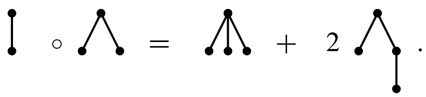
$$|t_1 \dots t_k| = |t_1| + \cdots + |t_k|,$$

where $|t|$ is the number of vertices of the rooted tree t, then $B_+ : S(k\mathcal{T}) \to k\mathcal{T}$ is an isomorphism of graded vector spaces.

There is a product $\circ$ on $k\mathcal{T}$ defined by Grossman and Larson [13]. Given rooted trees t and t', let $t = B_+(t_1 t_2 \dots t_n)$ and $|t'| = m$. Then $t \circ t'$ is the sum of the m^n rooted trees obtained by attaching each of the t_i to a vertex of t': if $t = \bullet$, set $t \circ t' = t'$. For example,



while



This noncommutative product makes $k\mathcal{T}$ a graded algebra with two-sided unit $\bullet$. There is a coproduct Δ on $k\mathcal{T}$ defined by $\Delta(\bullet) = \bullet \otimes \bullet$ and

$$\Delta(B_+(t_1 t_2 \dots t_k)) = \sum_{I \cup J = \{1,2,\dots,k\}} B_+(t(I)) \otimes B_+(t(J)), \tag{4.1}$$

where $t_1, \dots, t_k$ are rooted trees, the sum is over all disjoint pairs (I, J) of subsets of $\{1, 2, \dots, k\}$ such that $I \cup J = \{1, 2, \dots, k\}$, and $t(I)$ means the product of t_i for $i \in I$ (with the convention $B_+(t(\emptyset)) = \bullet$). As is proved in [13], the vector space $k\mathcal{T}$ with product $\circ$ and coproduct Δ is a graded connected Hopf algebra.

The Connes–Kreimer Hopf algebra $\mathcal{H}_K$ is generated as a commutative algebra by the rooted trees. As a graded algebra, $\mathcal{H}_K$ is $S(k\mathcal{T})$ with the grading discussed above. The coproduct on $\mathcal{H}_K$ can be described recursively by setting $\Delta(1) = 1 \otimes 1$ and

$$\Delta(t) = t \otimes 1 + (\mathrm{id} \otimes B_+)\Delta(B_-(t)), \tag{4.2}$$

for rooted trees t, where B_- is the inverse of B_+ and it is assumed that Δ acts multiplicatively on products of rooted trees.

Alternatively, Δ can be described on rooted trees t by the formula

$$\Delta(t) = t \otimes 1 + \sum_{\text{admissible cuts } c} P^c(t) \otimes R^c(t). \tag{4.3}$$

Here a cut of a rooted tree t is a subset of the edges of t, and a cut c is admissible if any path from the root to a leaf meets c at most once. If all the edges in c are removed from t, then t falls apart into smaller rooted trees: $R^c(t)$ is the component containing the original root, and $P^c(t)$ is the forest consisting of the rest of the components. We then extend Δ to forests by assuming it acts multiplicatively.

There is also a nice formula for the antipode of $\mathcal{H}_K$ in terms of cuts: for a rooted tree t,

$$S(t) = -\sum_{\text{all cuts } c} (-1)^{|c|} P^c(t) R^c(t), \tag{4.4}$$

where the sum is over all cuts c, and $|c|$ is the number of edges in c. Equation (4.4) can be proved by induction on $|t|$.

It follows from equation (4.3) that the "ladders" ℓ_i (where ℓ_i is the unbranched tree with i vertices) are divided powers. Thus, there is a Hopf algebra homomorphism ϕ: Sym $\to \mathcal{H}_K$ sending e_i to ℓ_i.

We note that, for a forest u and rooted tree t, the rooted tree t' can only appear in $B_+(u) \circ t'$ if there is an admissible cut c of t' such that

$$P^c(t') = u \quad \text{and} \quad R^c(t') = t. \tag{4.5}$$

Generalizing the definition of [14, §4], we define

$$n(u, t; t') = \text{the number of times } t' \text{ appears in } B_+(u) \circ t$$

and

$$m(u, t; t') = \text{the number of distinct admissible cuts } c \text{ for which (4.5) holds.}$$

Lemma 4.1. *For u, t, t' as above,*

$$n(u, t; t')|\operatorname{Symm}(t')| = m(u, t; t')|\operatorname{Symm}(B_+(u))||\operatorname{Symm}(t)|.$$

Proof. This is a slight extension of the proof of [14, Prop. 4.3]. First, let $u = s_1 s_2 \dots s_n$ for rooted trees s_i: then

$$\operatorname{Symm}(B_+(u)) = P \times \prod_{i=1}^{n} \operatorname{Symm}(s_i),$$

where P is the group that permutes those s_i that are isomorphic. Suppose (4.5) holds: let $\operatorname{Fix}(c, t')$ be the subgroup of $\operatorname{Symm}(t')$ that holds all the edges of c and everything "below" them pointwise fixed, and Q the subgroup of P that permutes identical parts of u that are attached to the same vertex in t'. Then $m(u, t; t')$ is the cardinality of the orbit of c under $\operatorname{Symm}(t')$, which is

$$\operatorname{Symm}(t')/\operatorname{Fix}(c, t') \times \prod_{i=1}^{n} \operatorname{Symm}(s_i) \times Q,$$

and so

$$m(u, t; t') = \frac{|\operatorname{Symm}(t')|}{|\operatorname{Fix}(c, t')||Q| \prod_{i=1}^{n} |\operatorname{Symm}(s_i)|}.$$

On the other hand, if we think of attaching the parts of u to the rooted tree t, we see that

$$n(u,t;t') = \frac{|P||\operatorname{Symm}(t)|}{|Q||\operatorname{Fix}(c,t')|}$$

and the conclusion follows. □

Using the lemma, we can prove that the Hopf algebras $\mathcal{H}_K$ and $k\mathcal{T}$ are dual to each other: for a somewhat different proof, see [14, Prop. 4.4].

Theorem 4.2. *The Hopf algebra $k\mathcal{T}$ is the graded dual of $\mathcal{H}_K$.*

Proof. First note that there is an inner product on $k\mathcal{T}$ defined by

$$(t,t') = \begin{cases} |\operatorname{Symm}(t)| & \text{if } t' = t, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

This inner product extends to $S(k\mathcal{T}) = \mathcal{H}_K$ via $(u,v) = (B_+(u), B_+(v))$ (since $\operatorname{Symm}(B_+(t)) \cong \operatorname{Symm}(t)$, this definition is consistent). So if we use Theorem 2.1 with $\phi = B_+$, hypothesis (a) of the theorem is satisfied. Hypothesis (b) follows easily from definitions, so it remains to prove

$$(u \otimes v, \Delta(w)) = (B_+(u) \circ B_+(v), B_+(w)) \tag{4.7}$$

for monomials u, v, and w of $\mathcal{H}_K$. Writing $t_1 = B_+(u)$, $t_2 = B_+(v)$ and $t_3 = B_+(w)$, equation (4.7) is

$$(B_-(t_1) \otimes B_-(t_2), \Delta(B_-(t_3))) = (t_1 \circ t_2, t_3),$$

which in turn, by using equation (4.2), is

$$(u \otimes t_2, \Delta(t_3) - t_3 \otimes 1) = (t_1 \circ t_2, t_3). \tag{4.8}$$

Both sides of equation (4.8) are nonzero if and only if there is an admissible cut c of t_3 such that

$$P^c(t_3) = u \quad \text{and} \quad R^c(t_3) = t_2,$$

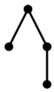
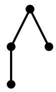
in which case it is

$$m(u,t_2;t_3)|\operatorname{Symm}(t_1)||\operatorname{Symm}(t_2)| = n(u,t_2;t_3)|\operatorname{Symm}(t_3)|,$$

i.e., the lemma above. □

5 Hopf algebras of planar rooted trees

In parallel to the preceding section, we define $\mathcal{P}$ to be the graded poset of planar rooted trees, and $k\mathcal{P}$ the corresponding graded vector space. A planar rooted tree is a particular realization of a rooted tree in the plane, so we consider

 and 

as distinct planar rooted trees. The tensor algebra $T(k\mathcal{P})$ can be regarded as the algebra of ordered forests of planar rooted trees, and there is a linear map $B_+ : T(k\mathcal{P}) \to k\mathcal{P}$ that makes a planar rooted tree out of an ordered forest of planar rooted trees by attaching a new root vertex. With the same conventions about grading as in the previous section, B_+ is an isomorphism of graded vector spaces.

Planar rooted trees with n non-root vertices correspond to balanced bracket arrangements (BBAs) of weight n, i.e., arrangements of the symbols $\langle$ and $\rangle$ such that

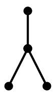
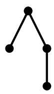
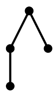
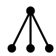
(1) the symbol $\langle$ and the symbol $\rangle$ each occur n times, and

(2) reading left to right, the count of $\langle$'s never falls behind the count of $\rangle$'s.

For example, the five BBAs of weight 3, to wit

$$\langle\langle\langle\rangle\rangle\rangle, \quad \langle\langle\rangle\langle\rangle\rangle, \quad \langle\rangle\langle\langle\rangle\rangle, \quad \langle\langle\rangle\rangle\langle\rangle, \quad \text{and} \quad \langle\rangle\langle\rangle\langle\rangle,$$

correspond respectively to

 ,  ,  ,  , and 

in $\mathcal{P}_3$. Note that the empty BBA corresponds to the 1-vertex tree $\bullet$. This representation is similar to that of Holtkamp [15], but differs in that our BBAs are not necessarily irreducible (see the next paragraph): to go from Holtkamp's representation to ours, remove the outermost pair of brackets. It is well known that the number of BBAs of weight n is the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

We call a BBA c irreducible if $c = \langle c' \rangle$ for some BBA c'. If a BBA is not irreducible, it can be written as a juxtaposition $c_1 c_2 \dots c_k$ of irreducible BBAs, which we call the components of c. The components of a BBA correspond to the branches of the root in the associated planar rooted tree.

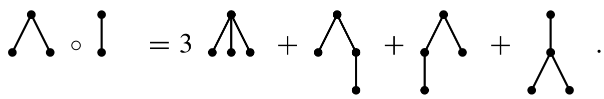
We define a product on $k\mathcal{P}$ via the representation in terms of BBAs. If the planar rooted trees T and T' are represented by BBAs c and c' respectively, let $c_1 c_2 \dots c_k$ be the components of c. Then $T \circ T'$ is the sum of planar rooted trees corresponding to the asymmetric shuffle product of c with c', i.e., the sum of the BBAs obtained by shuffling the symbols $c_1 c_2 \dots c_k$ into the BBA c'. For example, if $c = c_1 c_2$ and $c' = \langle\rangle$ then the asymmetric shuffle product $c \sqcup\!\sqcup c'$ is

$$c_1 c_2 \langle\rangle + c_1 \langle c_2 \rangle + c_1 \langle\rangle c_2 + \langle c_1 c_2 \rangle + \langle c_1 \rangle c_2 + \langle\rangle c_1 c_2.$$

If $c_1 = c_2 = \langle\rangle$, this reduces to

$$\langle\rangle\langle\rangle \sqcup\!\sqcup \langle\rangle = 3\langle\rangle\langle\rangle\langle\rangle + \langle\rangle\langle\langle\rangle\rangle + \langle\langle\rangle\rangle\langle\rangle + \langle\langle\rangle\langle\rangle\rangle,$$

and hence



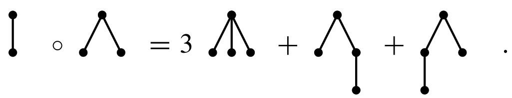
On the other hand, shuffling a single component c into $\langle\rangle\langle\rangle$ gives

$$c\langle\rangle\langle\rangle + \langle c\rangle\langle\rangle + \langle\rangle c\langle\rangle + \langle\rangle\langle c\rangle + \langle\rangle\langle\rangle c,$$

which for $c = \langle\rangle$ reduces to

$$\langle\rangle \sqcup\!\sqcup \langle\rangle\langle\rangle = 3\langle\rangle\langle\rangle\langle\rangle + \langle\rangle\langle\langle\rangle\rangle + \langle\langle\rangle\rangle\langle\rangle.$$

Thus



Now we make $k\mathcal{P}$ a coalgebra by defining a coproduct Δ on BBAs by

$$\Delta(c) = \sum_{i=0}^{k} c_1 \dots c_i \otimes c_{i+1} \dots c_k,$$

where $c = c_1 c_2 \dots c_k$ is the decomposition of c into irreducible components.

Theorem 5.1. *The product $\circ$ and coproduct Δ make $k\mathcal{P}$ a graded connected Hopf algebra.*

Proof. There are two main items to check: the associativity of $\circ$, and the multiplicativity of Δ. We use the representation of planar rooted trees by BBAs as outlined above. For BBAs a and b, each term of $a \sqcup\!\sqcup b$ has components that are either components of a, or components of b into which some components of a may be inserted: and the order of the components among those of a and among those of b is preserved. Thus each component of a term of $(a \sqcup\!\sqcup b) \sqcup\!\sqcup c$ is a component of a, a component of b into which some components of a may be inserted, or a component of c into which components of a and components of b (possibly including some components of a) may be inserted: and the order of components among those of a, b, and c is preserved. But terms of $a \sqcup\!\sqcup (b \sqcup\!\sqcup c)$ can be described the same way.

For multiplicativity, let a, b be BBAs, with decomposition into components $a = a_1 \dots a_n$ and $b = b_1 \dots b_m$. Then each term $c' \otimes c''$ of $\Delta(a \sqcup\!\sqcup b)$ comes from the term

$$(a_1 \dots a_i \sqcup\!\sqcup b_1 \dots b_j) \otimes (a_{i+1} \dots a_n \sqcup\!\sqcup b_{j+1} \dots b_m)$$

of $\Delta(a) \sqcup\!\sqcup \Delta(b)$, where i, j are the largest integers such that the components a_i and b_j respectively occur in c'. □

The Foissy Hopf algebra $\mathcal{H}_F$ is defined as follows. As an algebra, it is the tensor algebra $T(k\mathcal{P})$. The coalgebra structure can be defined by the same equation (4.3) as for $\mathcal{H}_K$, except that rooted trees are replaced by planar rooted trees, and the forests are ordered. (We remark that there is a natural order on the vertices of a planar rooted tree, which means that for a cut c of a planar rooted tree T the forest $P^c(T)$ has a natural ordering.)

Alternatively, the coalgebra structure on $\mathcal{H}_F$ can be defined by

$$\Delta(F) = \sum_{F' \subseteq F} (F - F') \otimes F'$$

where the sum is over all rooted subforests F' of F: if $F = T_1 T_2 \dots T_n$, then a rooted subforest F' of F is a forest $T_1' T_2' \dots T_n'$ such that each T_i' is either a subtree of T_i that contains the root, or empty. For such F and F', set $F - F' = F_1 F_2 \dots F_n$, where

$$F_i = \begin{cases} P^c(T_i) & \text{if } T_i' \neq \emptyset \text{ and } R^c(T_i) = T_i', \\ T_i & \text{if } T_i' = \emptyset. \end{cases}$$

The equation (4.4) for the antipode in $\mathcal{H}_K$ almost works in $\mathcal{H}_F$, but must be slightly modified. For a planar rooted tree T,

$$S(T) = -\sum_{\text{all cuts } c \text{ of } T} (-1)^{|c|} \overline{P^c(T)} R^c(T),$$

where $\overline{F}$ denotes the reverse of the ordered forest F (cf. [7, Théorème 44]). Note that S is an antiautomorphism of the noncommutative algebra $\mathcal{H}_F$, and $S^2 \neq \mathrm{id}$.

Theorem 5.2. *The Hopf algebra $(k\mathcal{P}, \circ, \Delta)$ is dual to the Foissy Hopf algebra $\mathcal{H}_F$.*

Proof. As in the preceding section, this boils down to the identity

$$(u \otimes v, \Delta(w)) = (B_+(u) \circ B_+(v), B_+(w)),$$

where now the inner product is defined by $(T, T') = \delta_{T,T'}$. The proof is essentially the same as that for Theorem 4.2 above, but much easier since there are no symmetry groups to complicate things: for planar rooted trees T, T' and an ordered forest F of planar rooted trees, $(B_+(F) \circ T, T')$ is both the multiplicity of T' in $B_+(F) \circ T$ and the number of cuts of T' with $P^c(T') = F$ and $R^c(T') = T$. □

Theorem 5.3. *$\mathcal{H}_F$ is self-dual.*

Proof. This follows from the existence of an inner product $(\cdot,\cdot)_F$ on $\mathcal{H}_F$ with

$$(F_1 F_2, F_3)_F = (F_1 \otimes F_2, \Delta(F_3))_F$$

for ordered forests F_1, F_2, F_3. Such an inner product is constructed in [7, §6]. □

6 Lifting to the Foissy Hopf algebra

The "ladder" trees ℓ_i can be thought of as planar rooted trees, and since they are divided powers in $\mathcal{H}_F$ there is a Hopf algebra homomorphism $\Phi\colon \mathrm{NSym} \to \mathcal{H}_F$ sending E_i to ℓ_i. In fact, there is a commutative diagram of Hopf algebras

$$\begin{tikzcd} \mathrm{NSym} \arrow[r, "\Phi"] \arrow[d, "\tau"'] & \mathcal{H}_F \arrow[d, "\rho"] \\ \mathrm{Sym} \arrow[r, "\phi"'] & \mathcal{H}_K \end{tikzcd} \tag{6.1}$$

where the map $\rho\colon \mathcal{H}_F \to \mathcal{H}_K$ sends each planar rooted tree to the corresponding rooted tree, and forgets order in products. In the commutative diagram dual to (6.1), i.e.,

$$\begin{tikzcd} \mathrm{QSym} & k\mathcal{P} \arrow[l, "\Phi^*"'] \\ \mathrm{Sym} \arrow[u, "\tau^*"] & k\mathcal{T} \arrow[l, "\phi^*"] \arrow[u, "\rho^*"'] \end{tikzcd} \tag{6.2}$$

the maps can be described explicitly as follows. As noted earlier, τ^* is the inclusion. For a partition $\lambda = \lambda_1, \lambda_2, \dots, \lambda_k$, let

$$t_\lambda = B_+(\ell_{\lambda_1}\ell_{\lambda_2}\dots\ell_{\lambda_k}) \in \mathcal{T}.$$

Then for rooted trees t,

$$\phi^*(t) = \begin{cases} |\operatorname{Symm}(t_\lambda)| m_\lambda & \text{if } t = t_\lambda \text{ for some partition } \lambda; \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Of course

$$|\operatorname{Symm}(t_\lambda)| = m_1(\lambda)! m_2(\lambda)! \dots,$$

where $m_i(\lambda)$ is the multiplicity of i in λ. The formula "upstairs" is simpler: if for a composition $I = (i_1, i_2, \dots, i_k)$ we define the planar rooted tree

$$T_I = B_+(\ell_{i_1}\ell_{i_2}\dots\ell_{i_k}) \in \mathcal{P},$$

then

$$\Phi^*(T) = \begin{cases} M_I, & \text{if } T = T_I \text{ for some composition } I; \\ 0, & \text{otherwise.} \end{cases}$$

For a rooted tree t,

$$\rho^*(t) = |\operatorname{Symm}(t)| \sum_{T \in \rho^{-1}(t)} T.$$

6.1 Some particular families of rooted trees

It is often easier to establish properties of rooted trees by working "upstairs" in diagram (6.2) rather than directly. For example, the elements

$$\kappa_n = \sum_{t \in \mathcal{T}_n} \frac{t}{|\operatorname{Symm}(t)|} \in k\mathcal{T}$$

are most easily understood by considering their images

$$\rho^*(\kappa_n) = \sum_{T \in \mathcal{P}_n} T \in k\mathcal{P}.$$

In this way it can be seen easily that the κ_n form a set of divided powers, and that $\phi^*(\kappa_n) = h_n$. Recalling the identity (3.2) in Sym, define elements ε_n of $k\mathcal{T}$ inductively by

$$\varepsilon_n = \kappa_1 \circ \varepsilon_{n-1} - \kappa_2 \circ \varepsilon_{n-2} + \cdots + (-1)^{n-1}\kappa_n, \quad \varepsilon_0 = \bullet. \tag{6.4}$$

Proposition 6.1. *The elements ε_n satisfy*

(a) $\varepsilon_n = (-1)^n S(\kappa_n)$,

(b) $\phi^*(\varepsilon_n) = e_n$,

(c) $n!\varepsilon_n = t_{1^n}$, *where 1^n is a string of n ones.*

Proof. Part (a) follows from equation (6.4), and then part (b) follows by applying ϕ^*. To prove part (c), recall from Theorem 4.2 that

$$\langle \chi(t), F\rangle = (t, B_+(F)),$$

where $(\cdot,\cdot)$ is the inner product given by (4.6). Hence $\chi(\kappa_n)$ is the linear functional on $\mathcal{H}_K$ that sends every forest of degree n to 1 (and all other forests to 0). If F is a forest of weight n, it follows from equation (4.4) that $\chi(\kappa_n)$ is zero on any forest $S(F)$ except

$$F = \bullet\bullet\cdots\bullet,$$

on which it is $(-1)^n$. Hence

$$S(\kappa_n) = (-1)^n \frac{B_+(\bullet\bullet\cdots\bullet)}{|\operatorname{Symm}(B_+(\bullet\bullet\cdots\bullet))|} = (-1)^n \frac{t_{1^n}}{n!}. \qquad \square$$

Zhao [18] defines a homomorphism NSym $\to k\mathcal{T}$ sending E_n to ε_n; in view of the preceding result, it sends the noncommutative analogue $(-1)^n S(E_n)$ of the nth complete symmetric function to κ_n. There are several distinct analogues of the power-sum symmetric functions in NSym (see [10]): their images in $k\mathcal{T}$ under Zhao's homomorphism are described in [18, Theorem 4.6].

If we define an operator $\mathfrak{N}\colon k\mathcal{T} \to k\mathcal{T}$ by $\mathfrak{N}(t) = \ell_2 \circ t$, then

$$\mathfrak{N}^k(t) = \sum_{|t'|=|t|+k} n(t;t')t',$$

for some coefficients $n(t;t')$. Apply ϕ^* to get

$$e_1^k \phi^*(t) = \sum_{|t'|=|t|+k} n(t;t')\phi^*(t'). \tag{6.5}$$

Two special cases of this equation are of some interest. First, let $k = 1$: then $n(t;t') = n(\bullet, t; t')$ as defined in §4 and equation (6.5) implies

$$e_1 \phi^*(t_\lambda) = \sum_{|\mu|=|\lambda|+1} n(\bullet, t_\lambda; t_\mu)\phi^*(t_\mu)$$

for all partitions λ. Hence, using (6.3) and Lemma 4.1,

$$m(\bullet, t_\lambda; t_\mu) = \frac{|\operatorname{Symm}(t_\mu)|}{|\operatorname{Symm}(t_\lambda)|} n(\bullet, t_\lambda; t_\mu) = \text{coefficient of } m_\mu \text{ in } e_1 m_\lambda.$$

Second, suppose $t = \bullet$. Then equation (6.5) is

$$e_1^k = \sum_{|\lambda|=k} n(\bullet; t_\lambda)\phi^*(t_\lambda),$$

which, compared with

$$e_1^k = \sum_{|\lambda|=k} \binom{k}{\lambda} m_\lambda,$$

gives a formula for $n(\bullet; t_\lambda)$ (cf. equation (1) of [2]):

$$n(\bullet; t_\lambda) = \frac{1}{|\operatorname{Symm}(t_\lambda)|}\binom{|\lambda|}{\lambda}.$$

6.2 Combinatorial Dyson–Schwinger equations

We now illustrate the use of the map ρ in diagram (6.1) to solve the combinatorial Dyson–Schwinger equation

$$X = 1 + B_+(X^p), \tag{6.6}$$

where X is a formal sum of elements

$$X = 1 + x_1 + x_2 + \cdots \tag{6.7}$$

of $\mathcal{H}_K$, with x_i of degree i, and p is a real number. If we write $\bar{X} = x_1 + x_2 + \cdots$, equation (6.6) is

$$\bar{X} = B_+((1+\bar{X})^p) = B_+\left(1 + \binom{p}{1}\bar{X} + \binom{p}{2}\bar{X}^2 + \cdots\right),$$

where

$$\binom{p}{k} = \frac{p(p-1)\ldots(p-k+1)}{k!}.$$

Then

$$x_{n+1} = B_+\left(\left\{1 + \binom{p}{1}\bar{X} + \binom{p}{2}\bar{X}^2 + \cdots\right\}_n\right),$$

where $\{\cdot\}_n$ means degree-n part. Consider (6.6) as an equation in $\mathcal{H}_F$: if

$$1 + X_1 + X_2 + \cdots = 1 + \tilde{X}$$

is its solution, then $X_1 = B_+(1) = \bullet$, and

$$X_{n+1} = B_+\left(\binom{p}{1}\{\tilde{X}\}_n + \binom{p}{2}\{(\tilde{X})^2\}_n + \cdots\right)$$

for $n \geq 1$. Thus

$$X_{n+1} = \sum_{k\leq n}\binom{p}{k}B_+\Big(\sum_{n_1+\cdots+n_k=n} X_{n_1}\ldots X_{n_k}\Big), \tag{6.8}$$

where the inner sum is over length-k compositions of n. We claim that equation (6.8) has the solution

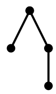
$$X_n = \sum_{T\in\mathcal{P}_{n-1}} C_p(T)T, \tag{6.9}$$

where $C_p(T)$ is defined as follows. For a vertex v of a planar rooted tree T, let $c(v)$ be the number of children of v. Let $\bar{V}(T)$ be the set of vertices of T with $c(v) \neq 0$: then

$$C_p(T) = \prod_{v\in\bar{V}(T)}\binom{p}{c(v)}. \tag{6.10}$$

For example,

$$C_p\Bigg(\qquad\Bigg) = \binom{p}{2}p.$$



To see that (6.9) really does solve equation (6.8), we use induction on n. Suppose equation (6.9) holds through dimension n, and consider the coefficient of T in X_{n+1} for $T \in \mathcal{P}_n$. Now T has a unique expression as $B_+(T_1T_2\ldots T_k)$, where $T_1T_2\ldots T_k$ is an ordered forest of planar rooted trees such that

$$|T_1| + |T_2| + \cdots + |T_k| = n.$$

From equation (6.8), we see that the only contribution to the coefficient of T can come from $X_{n_1}X_{n_2}\ldots X_{n_k}$, for $n_k = |T_k|$. By the induction hypothesis, the coefficient of T coming from equation (6.8) is

$$\binom{p}{k}C_p(T_1)C_p(T_2)\ldots C_p(T_k):$$

but this is evidently $C_p(T)$.

Now equation (6.10) makes sense for rooted trees t, and indeed for any planar rooted tree T we have $C_p(T) = C_p(\rho(T))$. Projecting $\tilde{X}$ to $\bar{X}$ via ρ gives the following result.

Theorem 6.2. *The solution* (6.7) *of the combinatorial Dyson–Schwinger equation* (6.6) *in* $\mathcal{H}_K$ *is*

$$x_n = \sum_{t \in \mathcal{T}_{n-1}} e(t) C_p(t) t,$$

where $e(t)$ *is the number of planar rooted trees* T *such that* $\rho(T) = t$.

To compare our result with that of [1, Lemma 4], note that

$$e(t) = \frac{1}{|\operatorname{Symm}(t)|} \prod_{v \in \bar{V}(t)} c(v)!,$$

so the coefficient in x_n of a rooted tree t is

$$\frac{1}{|\operatorname{Symm}(t)|} \prod_{v \in \bar{V}(t)} c(v)! \binom{p}{c(v)} = \frac{1}{|\operatorname{Symm}(t)|} \prod_{v \in \bar{V}(t)} p(p-1)\dots(p-c(v)+1).$$

The subalgebra of $\mathcal{H}_K$ generated by the x_n is in fact a sub-Hopf-algebra of $\mathcal{H}_K$. This follows from our final result, which gives an explicit formula for $\Delta(x_n)$ in terms of the x_i.

Theorem 6.3. *For the homogeneous parts* x_n *of the solution* (6.7) *of the combinatorial Dyson–Schwinger equation in* $\mathcal{H}_K$,

$$\Delta(x_n) = x_n \otimes 1 + \sum_{k=1}^{n} q_{n,k}(x_1, x_2, \dots) \otimes x_k, \tag{6.11}$$

where $q_{n,n} = 1$ *and*

$$q_{n,k}(x_1, x_2, \dots) = \sum_{i_1 + 2i_2 + \dots = n-k} \binom{k(p-1)+1}{i_1 + i_2 + \cdots} \frac{(i_1 + i_2 + \cdots)!}{i_1!\, i_2! \dots} x_1^{i_1} x_2^{i_2} \dots$$

for $1 \le k < n$.

Proof. We again work in $\mathcal{H}_F$ and project down to $\mathcal{H}_K$ via ρ. Equation (6.11) will follow from

$$\Delta(X_n) = X_n \otimes 1 + \sum_{k=1}^{n} Q_{n,k}(X_1, X_2, \dots) \otimes X_k, \tag{6.12}$$

where $Q_{n,n} = 1$ and

$$Q_{n,k}(X_1, X_2, \dots) = \sum_{q=1}^{n-k} \sum_{n_1 + \dots + n_q = n-k} \binom{k(p-1)+1}{q} X_{n_1} \dots X_{n_q}$$

for $1 \leq k < n$. To prove equation (6.12), we use the equation (4.3) for the coproduct in $\mathcal{H}_F$. Fix positive integers $k \leq n$ and consider those terms in $\Delta(X_n)$ that contribute to the term

$$C_p(T_1)T_1 \dots C_p(T_q)T_q \otimes C_p(T')T' \quad \text{in} \quad X_{|T_1|}X_{|T_2|} \dots X_{|T_q|} \otimes X_k \tag{6.13}$$

for particular planar rooted trees $T_1, \dots, T_q, T'$ such that $|T'| = k$ and $|T_1| + \dots + |T_q| = n - k$. They correspond to pairs (T, c), where T is a planar rooted tree of degree n and c is a cut of T such that

$$P^c(T) = T_1T_2 \dots T_q \quad \text{and} \quad R^c(T) = T'.$$

Then T is obtained by attaching $T_1, T_2, \dots, T_q$ (in order) to the vertices of T'. Let n_i be the number of the T_j attached to the ith vertex of T': then

$$\frac{C_p(T)}{C_p(T_1) \dots C_p(T_q)C_p(T')} = \prod_{i=1}^{k} \frac{\binom{p}{c_i + n_i}}{\binom{p}{c_i}} = \prod_{i=1}^{k} \frac{\binom{p - c_i}{n_i}}{\binom{c_i + n_i}{c_i}}, \tag{6.14}$$

where c_i is the number of children of the ith vertex of T'.

Now if we consider all the ways of attaching $T_1, \dots, T_q$ to the vertices of T' so that n_i of them are attached to the ith vertex of T', there are

$$\prod_{i=1}^{k} \binom{c_i + n_i}{c_i}$$

different configurations: they will generally be distinct as planar rooted trees, but the ratio (6.14) comes out the same. Thus, the sum of (6.14) over all the ways of doing the attachments is

$$\sum_{n_1 + \dots + n_q = q} \prod_{i=1}^{k} \binom{p - c_i}{n_i}$$

which by the generalized Vandermonde convolution (see [12, p. 248]) equals

$$\binom{kp - c_1 - \dots - c_k}{q} = \binom{kp - (k-1)}{q}$$

since the tree T' has a total of $k - 1$ edges. But this means that the sum of contributions from terms of $\Delta(X_n)$ to the coefficient of (6.13) is

$$\binom{k(p-1)+1}{q}. \qquad \square$$

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Formal differential equations and renormalization

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Abstract. The study of solutions of differential equations (analytic or formal) can often be reduced to a conjugacy problem, namely the conjugation of a given equation to a much simpler one, using identity-tangent diffeomorphisms.

On one hand, following Écalle's work (with a different terminology), such diffeomorphisms are given by characters on a given Hopf algebra (here a shuffle Hopf algebra). On the other hand, for some equations, the obstacles in the formal conjugacy are reflected in the fact that the associated characters appear to be ill-defined.

The analogy with the need for a renormalization scheme (dimensional regularization, Birkhoff decomposition) in quantum field theory becomes obvious for such equations and deliver a wide range of toy models. We discuss here the case of a simple class of differential equations where a renormalization scheme yields meaningful results.

1 Introduction

Let us start by giving a very simple example of a differential equation that already contains all the ingredients relevant to renormalization.

1.1 A toy model for some differential equations

Let us consider the equation

$$(E_{\alpha,d}) \qquad x^{1-d}\partial_x y = \alpha y^2,$$

where $d \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. Considering the right-hand side of this equation as a perturbation of the case $\alpha = 0$, we deal with the following conjugacy problem $(P_{\alpha,d})$: Does there exist a formal identity-tangent diffeomorphism

$$\Phi_{\alpha,d}(x,z) = (x, \varphi_{\alpha,d}(x,z)) \quad \varphi_{\alpha,d}(x,z) \in z + z^2\mathbb{C}[\![x,z]\!] \tag{1.1}$$

such that, if z is a solution of

$$(E_{0,d}) \qquad x^{1-d}\partial_x z = 0, \tag{1.2}$$

then $y = \varphi_{\alpha,d}(x,z)$ is a solution of $(E_{\alpha,d})$? Note that, in the sequel, we will always deal with diffeomorphisms that leave the x-coordinate unchanged (as $\Phi_{\alpha,d}(x,z)$) so we will focus on their nontrivial part: identity-tangent diffeomorphisms of the second variable whose coefficients depend on the first coordinate (as $\varphi_{\alpha,d}$).

It is quite obvious to check that here

$$\varphi_{\alpha,d}(x,z) = \frac{z}{1-\alpha\frac{x^d}{d}z} \tag{1.3}$$

is a very natural solution which, unfortunately, is ill-defined for $d = 0$ and, in this singular case, we are led to introduce the logarithm of x so that a good candidate for the conjugacy is

$$\varphi_{\alpha,0}(x,z) = \frac{z}{1-\alpha z\log x}$$

which is no more a formal series in x and z but is connected to the regular case ($d \geq 1$) in the following way:

1. The equation $(E_{\alpha,d})$ can be solved for any $d \in \mathbb{Z}^*$, assuming that the conjugating map $\varphi_{\alpha,d}$ has coefficients in $\mathbb{C}[\![x^d]\!]$, and even for $d \in \mathbb{R}^*$, assuming that we work with "ramified" powers of x.
2. When d is close to zero, writing $x^d = \sum_{n\geq 0}\frac{d^n\log^n x}{n!}$, the coefficients of $\varphi_{\alpha,d}$ appear as Laurent series in d.
3. One can then perform a Birkhoff decomposition of $\Phi_{\alpha,d}$: $\Phi_{\alpha,d} = \Phi^+_{\alpha,d}\circ\Phi^-_{\alpha,d}$ with
$$\Phi^+_{\alpha,d}(x,z) = \left(x, \frac{z}{1-\alpha\frac{(x^d-1)}{d}z}\right), \quad \Phi^-_{\alpha,d}(x,z) = \left(x, \frac{z}{1-\alpha\frac{1}{d}z}\right). \tag{1.4}$$
4. Since $\Phi^-_{\alpha,d}$ conjugates $(E_{0,d})$ to itself, $\Phi^+_{\alpha,d}$ also conjugates $(E_{0,d})$ to $(E_{\alpha,d})$ and when d goes to 0,
$$\lim_{d\to 0}\Phi^+_{\alpha,d}(x,z) = \left(x, \frac{z}{1-\alpha z\log x}\right) \tag{1.5}$$
conjugates $(E_{0,0})$ to $(E_{\alpha,0})$.

As we shall see now, this phenomenon can be generalized.

1.2 A generalization

Let $b(x,y) \in y^2\mathbb{C}[\![x,y]\!]$ and $d \in \mathbb{N}$. We will work on the following problem of formal conjugacy: does there exist a formal identity tangent diffeomorphism $\varphi(x,y)$ in y, with coefficients in $\mathfrak{A} = \mathbb{C}[\![x]\!]$ such that, if y is a solution of

$$(E_{b,d}) \qquad x^{1-d}\partial_x y = b(x,y), \tag{1.6}$$

then $z = \varphi(x,y)$ is a solution of

$$(E_{0,d}) \qquad x^{1-d}\partial_x z = 0. \tag{1.7}$$

As we shall see in Section 3, the answer is yes if $d \geq 1$, but rather than computing directly the coefficients of such a diffeomorphism, we will make an extensive use of Écalle's mould–comould expansions (see [6]). As we shall see in Section 2, the computation of such a diffeomorphism reduces to the computation of a character in a shuffle Hopf algebra.

In the case $d = 0$, this character happens to be ill-defined but the "dimensional regularization" suggested by the previous example gives us the final ingredient in order to perform a renormalization scheme that follows the same algebraic ideas developed in [2].

In order to introduce this scheme, let us first remark that computing an identity-tangent diffeomorphism is the same as computing a character in the Faà di Bruno Hopf algebra (see [7]).

1.3 Identity-tangent diffeomorphisms and character in the Faà di Bruno Hopf algebra

Let us consider the group of formal identity-tangent diffeomorphisms in one variable y, whose coefficients are in a commutative $\mathbb{C}$-algebra $\mathfrak{A}$:

$$G_{\mathfrak{A}} = \left\{f(y) = y + \textstyle\sum_{n\geq 1} f_n y^{n+1} \in \mathfrak{A}[\![y]\!]\right\}$$

with the product $\mu\colon G_{\mathfrak{A}} \times G_{\mathfrak{A}} \to G_{\mathfrak{A}}$ given by

$$\mu(f, g) = f \circ g.$$

For $n \geq 0$, the functionals on $G_{\mathfrak{A}}$ defined by

$$a_n(f) = \frac{1}{(n+1)!}(\partial_y^{n+1} f)(0) = f_n, \quad a_n\colon G_{\mathfrak{A}} \to \mathfrak{A}$$

are called the Faà di Bruno coordinates on the group $G_{\mathfrak{A}}$ and, $a_0 = 1$ being the unit, they generate a graded unital commutative algebra

$$\mathcal{H}_{\mathrm{FdB}} = \mathbb{C}[a_1, \ldots, a_n, \ldots] \quad (\mathrm{gr}(a_n) = n).$$

Moreover, the action of these functionals on a product in $G_{\mathfrak{A}}$ defines a coproduct on $\mathcal{H}_{\mathrm{FdB}}$ that turns to be a graded connected Hopf algebra (see [7] for details). For $n \geq 0$, the coproduct is defined by

$$a_n \circ \mu = m \circ \Delta(a_n), \tag{1.8}$$

where m is the usual multiplication in $\mathfrak{A}$, and the antipode reads

$$S \circ a_n = a_n \circ \mathrm{rec},$$

where $\mathrm{rec}(\varphi) = \varphi^{-1}$ is the composition inverse of φ.

Note that we can forget that the Faà di Bruno coordinates are functionals and then the Hopf algebra structure $\mathcal{H}_{\mathrm{FdB}}$ does not depend on the algebra $\mathfrak{A}$. Once we have such a Hopf algebra $\mathcal{H}$, one can define the group of characters on $\mathcal{H}$ with values in a commutative unital algebra $\mathfrak{A}$, that is to say algebra morphisms from $\mathcal{H}$ to $\mathfrak{A}$ and $\mathcal{C}(\mathcal{H}, \mathfrak{A})$ with the product

$$\varphi * \psi = m \circ (\varphi \otimes \psi) \circ \Delta \quad \text{for all } \varphi, \psi \in \mathcal{C}(\mathcal{H}, \mathfrak{A}). \tag{1.9}$$

This group is obviously isomorphic to $G_{\mathfrak{A}}$ so that computing some identity-tangent diffeomorphism means computing a character on the Faà di Bruno Hopf algebra and some renormalization scheme, if needed, can be used as in quantum field theory.

1.4 The renormalization scheme

We will now describe at an abstract level (assuming that the reader is familiar with graded Hopf algebras) what could be called a renormalization scheme in a Hopf algebra and, at each step, we will translate our first problem in terms of this scheme. Let us consider a mathematical problem (P) with the following properties:

Dimension parameter. The problem depends on a parameter $d \in \mathbb{N}$ (or $d \in \mathbb{Z}$ or else...) that can be called the dimension: $(P) = (P_d)$.

In the former conjugacy problem, d is obviously defined.

Hopf background. In the course of computing a solution to the problem (P_d), it appears that, using for example some perturbative expansions in some parameters other than d, we have to compute coefficients, with values in a commutative algebra $\mathfrak{A}$, indexed by a linear basis of a graded Hopf algebra $\mathcal{H}$ with product m and coproduct Δ (see Section 4 for an example of such a Hopf algebra). Moreover, if such coefficients exists, they define an element of the group of characters on $\mathcal{H}$ with values in $\mathfrak{A}$ ($\mathcal{C}(\mathcal{H}, \mathfrak{A})$).

In the former problem, one looks for a formal identity-tangent diffeomorphism $\varphi_{\alpha,d}$ in the variable z, that is to say a character on the Faà di Bruno Hopf algebra of coordinates of identity-tangent diffeomorphisms with values in $\mathfrak{A} = \mathbb{C}[\![x]\!]$.

Ill-defined character. Unfortunately, this character is ill-defined for some singular value d_0 of the dimension parameter.

In our example, this happens for $d = 0$.

Dimensional regularization. Working eventually in some extension $\mathfrak{B}$ of the algebra $\mathfrak{A}$, there is a way to generalize our problem to complex values of $d = d_0 + \varepsilon$ such that if $\varepsilon \neq 0$, one can compute a character ψ_ε with values in $\mathfrak{B}[\![\varepsilon]\!][\varepsilon^{-1}]$ (Laurent series with coefficients in $\mathfrak{B}$). Moreover, this gives the attempted character if $d_0 + \varepsilon = d$ is not a singular value of the parameter.

In our case, we have introduce a ramified power $x^\varepsilon = \exp(\varepsilon \log x)$ such that we can define an equation $(E_{\alpha,\varepsilon})$ and its associated character has its values in $\mathfrak{B}[\![\varepsilon]\!][\varepsilon^{-1}]$, with $\mathfrak{B} = \mathbb{C}[\![x, \log x]\!]$. The character ψ_ε corresponds to the diffeomorphism $\varphi_{\alpha,\varepsilon}$.

Birkhoff decomposition. As $\mathfrak{B}[\![\varepsilon]\!][\varepsilon^{-1}]$ is a Rota–Baxter algebra (see [5], [4]) with respect to the decomposition $\mathfrak{B}[\![\varepsilon]\!][\varepsilon^{-1}] = \varepsilon^{-1}\mathfrak{B}[\varepsilon^{-1}] \oplus \mathfrak{B}[\![\varepsilon]\!]$, there exists unique

Birkhoff decomposition of our character $\psi_\varepsilon = \psi_\varepsilon^+ * \psi_\varepsilon^-$ (or, depending of its pertinence in the problem, $\psi_\varepsilon = \psi_\varepsilon^- * \psi_\varepsilon^+$). And now $\psi_0^+ = \lim_{\varepsilon\to 0} \psi_\varepsilon^+$ is the renormalized value at $d = d_0$, for the given Hopf algebra and dimensional regularization.

In our first example, the Birkhoff-decomposition of the character is expressed on diffeomorphisms by $\Phi_{\alpha,\varepsilon} = \Phi_{\alpha,\varepsilon}^+ \circ \Phi_{\alpha,\varepsilon}^-$ and, thanks to the choice of the Hopf algebra and of the dimensional regularization, the renormalized character ψ_0^+ is a solution to the problem for the singular dimension $d = 0$. For details on renormalization and Hopf algebras, see [2], [3], [7].

1.5 Contents

In Section 2, we explain how conjugating diffeomorphisms for the equation $(E_{b,d})$ can be computed in a very algebraic way, using Écalle's mould–comould expansions. This gives the attempted result for $d \geq 1$ in Section 3.

In the case $d = 0$, we show how the previous computations yield an ill-defined character of a shuffle Hopf algebra. Using a quite natural dimensional regularization, this character can be renormalized (see Section 4).

We give then in Section 5, an interesting interpretation of the renormalized character for our conjugacy problem.

2 Mould–comould expansions and conjugacy of differential equations

Let us go back to the study of the equation

$$(E_{b,d}) \qquad x^{1-d}\partial_x y = b(x,y), \tag{2.1}$$

where $b(x,y) \in y^2\mathbb{C}[\![x,y]\!]$ and $d \in \mathbb{N}$. To compute the diffeomorphism (in the variable y) φ such that $z = \varphi(x,y)$ is a solution of

$$(E_{0,d}) \qquad x^{1-d}\partial_x z = 0 \tag{2.2}$$

we could try to compute its coefficients and thus, work in the Faà di Bruno Hopf algebra. As we shall see now, these computations are simpler and explicit when working with mould–comould expansions.

2.1 Diffeomorphisms and substitutions automorphisms

We are looking for identity-tangent diffeomorphisms

$$\varphi \in G_{\mathfrak{A}} = \{\varphi(x,y) \in y + y^2\mathfrak{A}[\![y]\!]\}.$$

Such a diffeomorphism defines a substitution automorphism on $\mathfrak{A}[\![y]\!]$:

$$F_\varphi(f) = f \circ \varphi \quad \text{for all } f \in \mathfrak{A}[\![y]\!],$$

such that $F_\varphi(fg) = F_\varphi(f)F_\varphi(g)$. Conversely, if F is an endomorphisms on $\mathfrak{A}[\![y]\!]$ such that $F(y) = \varphi(x, y) \in G_{\mathfrak{A}}$ and

$$F(fg) = F(f)F(g) \quad \text{for all } f, g \in \mathfrak{A}[\![y]\!],$$

then $F = F_\varphi$ (see [6]).

Moreover, using Taylor expansions, if $\varphi(y) = y + \sum_{n\geq 1} \varphi_n y^{n+1} \in G_{\mathfrak{A}}$, then

$$F_\varphi = \mathrm{Id} + \sum_{s\geq 1} \sum_{n_i \geq 1} \frac{1}{s!} \varphi_{n_1} \dots \varphi_{n_s} y^{n_1+\cdots+n_s+s} \partial_y^s \tag{2.3}$$

is a differential operator.

We will now look for substitutions automorphisms (rather than diffeomorphisms) that can be computed using elementary differential operators associated to the equation $(E_{b,d})$, that is to say mould–comould expansions.

2.2 Mould–comould expansions for the conjugacy problem

If y is a solution of the equation $(E_{b,d})$, then for any power series $f(y)$,

$$x^{1-d}\partial_x(f(y)) = x^{1-d}(\partial_x y)f'(y) = b(x, y)f'(y) = b(x, y)\partial_y f(y)$$

This suggests to consider the right-hand term of the equation $(E_{b,d})$ as a derivation. Using the expansion in x, we get

$$x^{1-d}\partial_x y = \sum_{n\geq 0} x^n b_n(y) = \sum_n x^n \mathbb{B}_n.y,$$

where

$$\mathbb{B}_n = b_n(y)\partial_y,$$

so that the data in $b(x, y)$ are encoded in the derivations $\mathbb{B}_n$. It seems reasonable to think that the conjugating diffeomorphism (or rather its associated substitution automorphism) can be expressed with the help of these operators. To do so, let

$$\mathcal{N} = \{\emptyset\} \cup \{\boldsymbol{n} = (n_1, \dots, n_s),\ s \geq 1,\ n_i \in \mathbb{N}\}$$

and

$$\mathbb{B}_{\boldsymbol{n}} = \mathbb{B}_{n_s} \dots \mathbb{B}_{n_1} \quad (\mathbb{B}_\emptyset = \mathrm{Id}). \tag{2.4}$$

Now that we have a set of differential operators, which is called a cosymmetral comould in Écalle's work (see [6]), this suggest that the attempted conjugating map $\varphi(x, y)$, or rather its associated substitution automorphism, may be expressed with the help of this “comould”:

$$F_\varphi = \mathrm{Id} + \sum_{s\geq 1}\sum_{n_1,\ldots,n_s\in\mathbb{N}} M^{n_1,\ldots,n_s}\mathbb{B}_{n_s}\ldots\mathbb{B}_{n_1} = \sum_{\boldsymbol{n}\in\mathcal{N}} M^{\boldsymbol{n}}\mathbb{B}_{\boldsymbol{n}} = \sum M^\bullet\mathbb{B}_\bullet,$$

where $M^\emptyset = 1$ (for identity-tangent diffeomorphism), $F_\varphi(y) = \varphi(x,y)$ and the collection of coefficients $M^\bullet$, which is called a mould, has its values in $\mathfrak{A} = \mathbb{C}[\![x]\!]$. In order to manipulate such mould–comould expansions, we will now give some classical results on moulds.

2.3 Reminder on moulds

For details see [6].

Definition 1. *A mould $M^\bullet$ on $\mathcal{N}$ with values in a commutative algebra $\mathfrak{A}$ is a map from $\mathcal{N}$ to $\mathfrak{A}$. Such a mould $M^\bullet$ is symmetral if $M^\emptyset = 1$ and*

$$M^{\boldsymbol{n}^1}M^{\boldsymbol{n}^2} = \sum_{\boldsymbol{n}\in\mathrm{sh}(\boldsymbol{n}^1,\boldsymbol{n}^2)} M^{\boldsymbol{n}} \quad \textit{for all } \boldsymbol{n}^1,\boldsymbol{n}^2\in\mathcal{N},$$

where the sum is over all the possible shuffling of the sequences $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$. A mould $M^\bullet$ is alternal if $M^\emptyset = 0$ and

$$\sum_{\boldsymbol{n}\in\mathrm{sh}(\boldsymbol{n}^1,\boldsymbol{n}^2)} M^{\boldsymbol{n}} = 0 \quad \textit{for all } \boldsymbol{n}^1,\boldsymbol{n}^2\in\mathcal{N}.$$

Provided that the series makes sense, to any mould $M^\bullet$ one can associate a differential operator

$$\mathbb{M} = \sum_{\boldsymbol{n}\in\mathcal{N}} M^{\boldsymbol{n}}\mathbb{B}_{\boldsymbol{n}} = \sum M^\bullet\mathbb{B}_\bullet.$$

For example,

$$b(x,y)\partial_y = \sum_n x^n\mathbb{B}_n = \sum_{\boldsymbol{n}\in\mathcal{N}} I^{\boldsymbol{n}}\mathbb{B}_{\boldsymbol{n}} = \sum I^\bullet\mathbb{B}_\bullet,$$

where $I^\emptyset = 0$ and

$$I^{n_1,\ldots,n_s} = \begin{cases} x^{n_1} & \text{if } s = 1,\\ 0 & \text{otherwise,}\end{cases}$$

defines an alternal mould.

If $M^\bullet$ and $N^\bullet$ are two moulds, then

$$\begin{aligned}
\mathbb{M}.\mathbb{N} &= \Big(\sum_{\boldsymbol{n}^1\in\mathcal{N}} M^{\boldsymbol{n}^1}\mathbb{B}_{\boldsymbol{n}^1}\Big).\Big(\sum_{\boldsymbol{n}^2\in\mathcal{N}} N^{\boldsymbol{n}^2}\mathbb{B}_{\boldsymbol{n}^2}\Big)\\
&= \sum_{\boldsymbol{n}^1,\boldsymbol{n}^2} M^{\boldsymbol{n}^1}N^{\boldsymbol{n}^2}\mathbb{B}_{\boldsymbol{n}^1}\mathbb{B}_{\boldsymbol{n}^2}\\
&= \sum_{\boldsymbol{n}^1,\boldsymbol{n}^2} M^{\boldsymbol{n}^1}N^{\boldsymbol{n}^2}\mathbb{B}_{\boldsymbol{n}^2\boldsymbol{n}^1} \quad \text{(see (2.4))}\\
&= \sum_{\boldsymbol{n}}\Big(\sum_{\boldsymbol{n}^2\boldsymbol{n}^1=\boldsymbol{n}} M^{\boldsymbol{n}^1}N^{\boldsymbol{n}^2}\Big)\mathbb{B}_{\boldsymbol{n}}\\
&= \sum_{\boldsymbol{n}}\Big(\sum_{\boldsymbol{n}^1\boldsymbol{n}^2=\boldsymbol{n}} N^{\boldsymbol{n}^1}M^{\boldsymbol{n}^2}\Big)\mathbb{B}_{\boldsymbol{n}},
\end{aligned}$$

where the sum is over pairs $(\boldsymbol{n}^1,\boldsymbol{n}^2)$ whose concatenation gives $\boldsymbol{n}$. These formulas define a product on moulds:

Proposition 1. *For any moulds $M^\bullet$ and $N^\bullet$, their product $P^\bullet = M^\bullet \times N^\bullet$ is defined by*

$$P^{\boldsymbol{n}} = \sum_{\boldsymbol{n}^1\boldsymbol{n}^2=\boldsymbol{n}} M^{\boldsymbol{n}^1}N^{\boldsymbol{n}^2} \quad \textit{for all } \boldsymbol{n}\in\mathcal{N}.$$

Moreover the set of symmetral moulds, is a group whose unit $1^\bullet$ is given by $1^\emptyset = 1$ and $1^{\boldsymbol{n}} = 0$ otherwise. The inverse $N^\bullet$ of a given symmetral mould $M^\bullet$ is given by $N^\emptyset = 1$ and

$$N^{n_1,\ldots,n_s} = (-1)^s M^{n_s,\ldots,n_1}.$$

Of course, specialists of Hopf algebras can already smell the flavor of a shuffle Hopf algebra here and we will see the connection in Section 4.

Symmetral moulds play a central role in the search of conjugating diffeomorphisms since

Proposition 2. *If $M^\bullet$ is a symmetral mould, then its associated mould–comould expansion $\mathbb{M}$ is a substitution automorphism corresponding to the diffeomorphism $m(x,y) = \mathbb{M}.y$. Moreover if $M^\bullet$ and $N^\bullet$ are two symmetral moulds corresponding to diffeomorphisms m and n, then the mould $P^\bullet = M^\bullet \times N^\bullet$ corresponds to the diffeomorphism $m\circ n$.*

For the first part of this proposition, see [6]. For the second part,

$$m\circ n(x,y) = \mathbb{N}.\mathbb{M}.y = \sum P^\bullet\mathbb{B}_\bullet y = \mathbb{P}.y.$$

With this short reminder on moulds, we are now ready to deal with the equation $(E_{b,d})$ when everything works, that is to say when $d \in \mathbb{N}^*$.

3 The case $d \in \mathbb{N}^*$

Suppose that $z = \varphi(x, y)$ conjugates the equation

$$(E_{b,d}) \qquad x^{1-d}\partial_x y = b(x, y)$$

to $x^{1-d}\partial_x z$=0. One expects that the associated substitution automorphism can be written as a mould–comould expansion

$$\varphi(x, y) = \sum V_d^\bullet \mathbb{B}_\bullet y = \mathbb{V}_d.y,$$

where $V_d^\bullet$ is a symmetral mould. The equation yields

$$\begin{aligned}
x^{1-d}\partial_x z &= x^{1-d}\partial_x\varphi(x, y)\\
&= \sum x^{1-d}\partial_x(V_d^\bullet \mathbb{B}_\bullet y)\\
&= \sum (x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \sum V_d^\bullet (x^{1-d}\partial_x(\mathbb{B}_\bullet y))\\
&= \sum (x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \sum V_d^\bullet (x^{1-d}\partial_x y)(\partial_y \mathbb{B}_\bullet y)\\
&= \sum (x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \sum V_d^\bullet b(x, y)(\partial_y \mathbb{B}_\bullet y)\\
&= \sum (x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \left(\sum I^\bullet \mathbb{B}_\bullet\right)\left(\sum V_d^\bullet \mathbb{B}_\bullet\right).y\\
&= \sum (x^{1-d}\partial_x V_d^\bullet)\mathbb{B}_\bullet y + \sum (V_d^\bullet \times I^\bullet)\mathbb{B}_\bullet y\\
&= 0.
\end{aligned}$$

This suggest to look for a symmetral mould $V_d^\bullet$ such that $V_d^\emptyset = 1$ and

$$x^{1-d}\partial_x V_d^\bullet = -V_d^\bullet \times I^\bullet. \tag{3.1}$$

Of course the conjugacy of $x^{1-d}\partial_x z = 0$ to $(E_{b,d})$ is given by the inverse of φ, which is given by the inverse of $V_d^\bullet$, namely $U_d^\bullet$, that satisfies the equation

$$x^{1-d}\partial_x U_d^\bullet = I^\bullet \times U_d^\bullet. \tag{3.2}$$

A straightforward computation shows that one can make the following choice:

Proposition 3. *For $d \geq 1$, the moulds defined for $(n_1, \ldots, n_s) \in \mathcal{N}$ by*

$$\begin{aligned}
U_d^{n_1,\ldots,n_s} &= \frac{x^{n_1+\cdots+n_s+sd}}{(\hat{n}_1 + sd)(\hat{n}_2 + (s-1)d)\ldots(\hat{n}_s + d)} \qquad (\hat{n}_i = n_i + \cdots + n_s),\\
V_d^{n_1,\ldots,n_s} &= \frac{(-1)^s x^{n_1+\cdots+n_s+sd}}{(\check{n}_1 + d)(\check{n}_2 + 2d)\ldots(\check{n}_s + sd)} \qquad (\check{n}_i = n_1 + \cdots + n_i)
\end{aligned}$$

are symmetral and solutions of the previous equations. Moreover the substitution automorphism defined by $U_d^\bullet$ (resp. $V_d^\bullet$) conjugates $(E_{0,d})$ to $(E_{b,d})$ (resp. $(E_{b,d})$ to $(E_{0,d})$).

Unfortunately, if $d = 0$, the mould $V_d^\bullet$ is ill-defined (for example if $n_1 = 0$). This really looks like the situation that occurs in quantum field theory and calls for some renormalization. We will now describe a renormalization scheme at $d = 0$.

4 Renormalization in a shuffle Hopf algebra

In order to use a renormalization scheme, we will first give the quite obvious Hopf algebra settings related to such symmetral moulds. We will then describe a very natural dimensional regularization and perform the renormalization in the "mould" terminology.

4.1 The shuffle Hopf algebra $\mathrm{sh}_{\mathcal{N}}$

Once again, let

$$\mathcal{N} = \{\emptyset\} \cup \{\boldsymbol{n} = (n_1, \ldots, n_s),\ s \geq 1,\ n_i \in \mathbb{N}\}.$$

If

$$l(n_1, \ldots, n_s) = s \quad (l(\emptyset) = 0), \quad \|(n_1, \ldots, n_s)\| = n_1 + \cdots + n_s \quad (\|\emptyset\| = 0),$$

then the linear span of $\mathcal{N}$ is a graded (for the graduation $\|.\| + l(.)$) vector space with finite dimensional graded components. This space $\mathrm{sh}_{\mathcal{N}}$ turns to be a Hopf algebra with the following definitions. The product is as follows:

- $\emptyset$ is the unit.
- For $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ in $\mathcal{N}$, the product $m\colon \mathrm{sh}_{\mathcal{N}} \otimes \mathrm{sh}_{\mathcal{N}} \to \mathrm{sh}_{\mathcal{N}}$ is defined by

$$m(\boldsymbol{n}^1 \otimes \boldsymbol{n}^2) = \sum_{\boldsymbol{n} \in \mathrm{sh}(\boldsymbol{n}^1, \boldsymbol{n}^2)} \boldsymbol{n},$$

  where the sum is over all the possible shuffling of the tuples $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$.

For example

$$m((n_1) \otimes (n_2, n_3)) = (n_1, n_2, n_3) + (n_2, n_1, n_3) + (n_2, n_3, n_1).$$

With this product, $\mathrm{sh}_{\mathcal{N}}$ is a graded commutative algebra and it remains to define the coproduct $\Delta\colon \mathrm{sh}_{\mathcal{N}} \to \mathrm{sh}_{\mathcal{N}} \otimes \mathrm{sh}_{\mathcal{N}}$:

- $\Delta\emptyset = \emptyset \otimes \emptyset$.
- For $\boldsymbol{n} \in \mathcal{N}$,

$$\Delta(\boldsymbol{n}) = \sum_{\boldsymbol{n} = \boldsymbol{n}^1 \boldsymbol{n}^2} \boldsymbol{n}^1 \otimes \boldsymbol{n}^2,$$

  where the sum is over the pairs $(\boldsymbol{n}^1, \boldsymbol{n}^2)$ whose concatenation gives $\boldsymbol{n}$.

For example,

$$\Delta(n_1,n_2,n_3)=(n_1,n_2,n_3)\otimes\emptyset+(n_1,n_2)\otimes(n_3)+(n_1)\otimes(n_2,n_3)+\emptyset\otimes(n_1,n_2,n_3).$$

With these product and coproduct, $\mathrm{sh}_{\mathcal{N}}$ is a very classical graded connected Hopf algebra whose antipode is given by

$$S(n_1,\ldots,n_s) = (-1)^s(n_s,\ldots,n_1).$$

For details on Hopf algebras and shuffle Hopf algebras, see [1].

When one deals with Hopf algebras, one can define characters and it is now obvious that symmetral moulds and characters are strongly connected: If $\mathfrak{A}$ is a commutative unital algebra then characters (algebra morphisms) on $\mathrm{sh}_{\mathcal{N}}$ with values in $\mathfrak{A}$ form a group for the product

$$\varphi * \psi = m \circ (\varphi \otimes \psi) \circ \Delta \quad \text{for all } \varphi, \psi \in \mathcal{C}(\mathrm{sh}_{\mathcal{N}}, \mathfrak{A}).$$

This group is isomorphic to Écalle's group of symmetral moulds with values in $\mathfrak{A}$ since a symmetral mould can be identified to the image of the basis $\mathcal{N}$ by a character.

Now we are ready to express a renormalization scheme on some examples.

4.2 Divergences for the moulds (or characters) $U_d^\bullet$ and $V_d^\bullet$

In our case $\mathfrak{A} = \mathbb{C}[\![x]\!]$ and, as quoted before, our "characters" $U_d^\bullet$ and $V_d^\bullet$ are unfortunately ill-defined when $d = 0$. When looking at $V_d^\bullet$, if for $(n_1,\ldots,n_s) \in \mathcal{N}$,

$$D(n_1,\ldots,n_s) = \max\{0 \le i \le s; \text{for all } 1 \le j \le i,\ \check{n}_j = 0\},$$

from the physicists point of view the following holds:

- if $D(n_1,\ldots,n_s) = 0$, $V_d^{n_1,\ldots,n_s}$ has no divergence at $d = 0$,
- if $D(n_1,\ldots,n_s) = 1$, $V_d^{n_1,\ldots,n_s}$ has an overall divergence but no subdivergence at $d = 0$,
- if $D(n_1,\ldots,n_s) > 1$, $V_d^{n_1,\ldots,n_s}$ has an overall divergence and $D(n_1,\ldots,n_s) - 1$ subdivergences at $d = 0$.

Now the formula for $V_d^\bullet$ suggest that we could define a dimensional regularization by using the same formula for $d = \varepsilon \in \mathbb{C}^*$. The price to pay is to consider now that

$$x^\varepsilon = \sum_{n\ge 0} \frac{\varepsilon^n}{n!} \log^n x$$

so that, for $\varepsilon \in \mathbb{C}^*$ close to zero, the mould $V_\varepsilon^\bullet$ has its values in $\mathcal{A} = \mathfrak{B}[\![\varepsilon]\!][\varepsilon^{-1}]$ where $\mathfrak{B} = \mathbb{C}[\![x, \log x]\!]$. Using the usual Birkhoff decomposition in terms of moulds, we get

Theorem 1. *There exists a unique pair of moulds* $(C_\varepsilon^\bullet, R_\varepsilon^\bullet)$ *such that*

$$R_\varepsilon^\bullet = C_\varepsilon^\bullet \times V_\varepsilon^\bullet,$$

where $C_\varepsilon^\bullet$ (*counterterms*) *is symmetral with values in* $\mathcal{A}_- = \varepsilon^{-1}\mathfrak{B}[\varepsilon^{-1}]$ *and* $R_\varepsilon^\bullet$ (*regularized*) *is symmetral with values in* $\mathcal{A}_+ = \mathfrak{B}[\![\varepsilon]\!]$. *Moreover*

$$C_\varepsilon^{n_1,\dots,n_s} = \begin{cases} \dfrac{1}{s!\varepsilon^s} & \textit{if } n_1 = \cdots = n_s = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

For a proof see Section 6. We have now a renormalization scheme for our problem but, as in quantum field theory, this would be useless if it had no meaning for our equations. It is indeed meaningful as we shall see now.

5 Interpretation of the renormalized mould $R_\varepsilon^\bullet$

5.1 Ramified conjugacy

On one hand, the ill-definedness of $V_d^\bullet$ at $d = 0$ suggest that the equations

$$x\partial_x y = b(x, y)$$

cannot be formally (with diffeomorphisms in $\mathbb{C}[\![x, y]\!]$) conjugated to the equation

$$x\partial_x z = 0.$$

On the other hand, we chose a quite natural dimensional regularization for our mould $V_d^\bullet$ since for $\varepsilon \in \mathbb{C}^*$, we still have the equation

$$x^{1-\varepsilon}\partial_x V_\varepsilon^\bullet = -V_\varepsilon^\bullet \times I^\bullet$$

but now as $R_\varepsilon^\bullet = C_\varepsilon^\bullet \times V_\varepsilon^\bullet$ and $C_\varepsilon^\bullet$ does not depend on x,

$$\begin{aligned} x^{1-\varepsilon}\partial_x R_\varepsilon^\bullet &= x^{1-\varepsilon}\partial_x (C_\varepsilon^\bullet \times V_\varepsilon^\bullet) \\ &= C_\varepsilon^\bullet \times (x^{1-\varepsilon}\partial_x V_\varepsilon^\bullet) \\ &= -C_\varepsilon^\bullet \times V_\varepsilon^\bullet \times I^\bullet \\ &= -R_\varepsilon^\bullet \times I^\bullet. \end{aligned}$$

The mould $R_\varepsilon^\bullet$ (as $V_\varepsilon^\bullet$) defines a diffeomorphism that also conjugates the equation

$$x^{1-\varepsilon}\partial_x y = b(x, y)$$

to $x^{1-\varepsilon}\partial_x z = 0$. The mould $R_\varepsilon^\bullet$ is regular at $\varepsilon = 0$, with a price to pay: it contains monomials in x and $\log x$. When ε goes to 0, we get:

Theorem 2. *There exists a "ramified" identity tangent diffeomorphism* $\varphi(x, y) \in y + y^2\mathbb{C}[\![x, \log x, y]\!]$ *that conjugates* $x\partial_x y = b(x, y)$ *to* $x\partial_x z = 0$.

The need for logarithms, as well as the ill-definedness of a "formal" conjugating diffeomorphism, suggest that, in the case $d = 0$, some part of the right-hand term of the equation $x\partial_x y = b(x, y)$ cannot be canceled by formal conjugacy: there should remain some formal “invariants“. The next natural question becomes: If one cannot formally conjugate to $x\partial_x z = 0$, what is the simplest equation to which one can conjugate ?

The following section gives a partial answer to this.

5.2 The logarithmic-alogarithmic factorization of $R_0^\bullet$ and its interpretation

As it shall be proved in Section 6, we have

Theorem 3. *The symmetral mould $R_0^\bullet$ admits the following factorization:*

$$R_0^\bullet = L^\bullet \times S^\bullet,$$

where

1. *$L^\bullet$ is a purely logarithmic symmetral mould defined for $(n_1, \dots, n_s) \in \mathcal{N}$ by*

$$L^{n_1,\dots,n_s} = \begin{cases} \dfrac{(-1)^s}{s!} \log^s x & \textit{if } n_1 = \dots = n_s = 0, \\ 0 & \textit{otherwise;} \end{cases}$$

2. *$S^\bullet$ is a symmetral mould with values in $\mathbb{C}[\![x]\!]$.*

We have then

$$\begin{aligned} x\partial_x R_0^\bullet &= x\partial_x (L^\bullet \times S^\bullet) \\ &= L^\bullet \times (x\partial_x S^\bullet) + (x\partial_x L^\bullet) \times S^\bullet \\ &= -R_0^\bullet \times I^\bullet \\ &= -L^\bullet \times S^\bullet \times I^\bullet, \end{aligned}$$

and if $-L^\bullet \times A^\bullet = x\partial_x L^\bullet$, then

$$x\partial_x S^\bullet + S^\bullet \times I^\bullet = A^\bullet \times S^\bullet.$$

A straightforward computation shows that the mould $A^\bullet$ is alternal ($A^\emptyset = 0$) and for $(n_1, \dots, n_s) \in \mathcal{N}$,

$$A^{n_1,\dots,n_s} = \begin{cases} 1 & \text{if } s = 1 \text{ and } n_1 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\sum A^\bullet \mathbb{B}_\bullet y = b(0, y).$$

But now, if φ^{nor} is the formal diffeomorphism associated to $S^\bullet$ and $z = \varphi^{\mathrm{nor}}(x, y)$ with

$$x\partial_x y = b(x, y),$$

then, as in the computations for $V_d^\bullet$,

$$\begin{aligned}
x\partial_x z &= x\partial_x \varphi^{\mathrm{nor}}(x,y)\\
&= x\partial_x\Big(\sum S^\bullet \mathbb{B}_\bullet y\Big)\\
&= \sum (x\partial_x S^\bullet + S^\bullet \times I^\bullet)\mathbb{B}_\bullet y\\
&= \sum (A^\bullet \times S^\bullet)\mathbb{B}_\bullet y\\
&= \Big(\sum S^\bullet \mathbb{B}_\bullet\Big)\Big(\sum A^\bullet \mathbb{B}_\bullet y\Big)\\
&= b(0,\varphi^{\mathrm{nor}}(x,y))\\
&= b(0,z).
\end{aligned}$$

Until $b(0,y)=0$, the previous results suggests that the equation $x\partial_x y = b(x,y)$ cannot be formally conjugated to $x\partial_x z = 0$ but, at least, it is formally conjugated to a "normal" equation

$$x\partial_x z = b(0,z) = b_0(z).$$

Moreover, it is easy to check that the only way to conjugate $x\partial_x z = b_0(z)$ to $x\partial_x z = 0$ is to use a diffeomorphism in $\mathbb{C}[\![\log x, z]\!]$, that corresponds to the mould $L^\bullet$ in the factorization of $R_0^\bullet$.

6 Proofs

It remains to prove that $R_\varepsilon^\bullet = C_\varepsilon^\bullet \times V_\varepsilon^\bullet$ with

$$C_\varepsilon^{n_1,\dots,n_s} = \begin{cases} \dfrac{1}{s!\varepsilon^s} & \text{if } n_1=\dots=n_s=0,\\ 0 & \text{otherwise,}\end{cases}$$

and $R_0^\bullet = L^\bullet \times S^\bullet$ with

$$L^{n_1,\dots,n_s} = \begin{cases} \dfrac{(-1)^s}{s!}\log^s x & \text{if } n_1=\dots=n_s=0,\\ 0 & \text{otherwise.}\end{cases}$$

Let us suppose that $C_\varepsilon^\bullet$ is defined as above. Since it is symmetral, with values in $\mathcal{A}_- = \varepsilon^{-1}\mathfrak{B}[\varepsilon^{-1}]$, it is clear that $R_\varepsilon^\bullet$ is symmetral and it remains to prove that $R_\varepsilon^\bullet$ has its values in $\mathcal{A}_+ = \mathfrak{B}[\![\varepsilon]\!]$. Let $0^{(k)}$ be the sequence with k zeros. Any non empty sequence in $\mathcal{N}$ can be written $\boldsymbol{n}^k = (0^{(k)}, n_1,\dots,n_s) = (0^{(k)}\boldsymbol{n})$ with $k\ge 0$, $s\ge 0$ and $\boldsymbol{n} = (n_1,\dots,n_s)$ is such that

$$\boldsymbol{n}=\emptyset \quad\text{or}\quad \boldsymbol{n}\neq\emptyset \quad\text{but}\quad n_1\neq 0.$$

It is clear now that

$$R_\varepsilon^{0^{(k)}\boldsymbol{n}} = (C_\varepsilon^\bullet \times V_\varepsilon^\bullet)^{0^{(k)}\boldsymbol{n}} = \sum_{j=0}^{k} \frac{1}{(k-j)!\varepsilon^{k-j}} V_\varepsilon^{0^{(j)}\boldsymbol{n}}.$$

Let us consider first the case $\boldsymbol{n} = \emptyset$, $k \geq 1$,

$$\begin{aligned} R_\varepsilon^{0^{(k)}} &= \sum_{j=0}^{k} \frac{1}{(k-j)!\varepsilon^{k-j}} V_\varepsilon^{0^{(j)}} \\ &= \frac{1}{\varepsilon^k} \sum_{j=0}^{k} \frac{1}{(k-j)!} \cdot \frac{(-1)^j x^{j\varepsilon}}{j!} \\ &= \frac{1}{k!} \left(\frac{1-x^\varepsilon}{\varepsilon} \right)^k . \end{aligned}$$

It is clear that, after expansion (in ε), this coefficient belongs to $\mathcal{A}_+$ and

$$R_0^{0^{(k)}} = \frac{(-\log x)^k}{k!} = (L^\bullet \times S^\bullet)^{0^{(k)}}$$

with $S^{0^{(j)}} = 0$ for $j \geq 1$.

Let us suppose now that $\boldsymbol{n} = (n_1, \ldots, n_s)$ is non empty and $n_1 \neq 0$. If $p_i = n_1 + \cdots + n_i$, then, after expansion in the variable ε, for $j \geq 0$

$$\begin{aligned} V_\varepsilon^{0^{(j)}\boldsymbol{n}} &= \frac{(-1)^{j+s} x^{n_1+\cdots+n_s+(s+j)\varepsilon}}{j!\varepsilon^j (p_1 + (j+1)\varepsilon) \ldots (p_s + (j+s)\varepsilon)} \\ &= \frac{(-1)^{j+s} x^{\|\boldsymbol{n}\|}}{j!\varepsilon^j} \sum_{\substack{l_t \geq 0 \\ 0 \leq t \leq s}} \frac{\varepsilon^{\|\boldsymbol{l}\|}(-1)^{\|\boldsymbol{l}\|-l_0}}{l_0! p_1^{l_1+1} \ldots p_s^{l_s+1}} (\log x)^{l_0} \gamma^{l_0,\ldots,l_s}(j), \end{aligned}$$

where $\|\boldsymbol{n}\| = n_1 + \cdots + n_s$, $\boldsymbol{l} = (l_0, \ldots, l_s)$, $\|\boldsymbol{l}\| = l_0 + \cdots + l_s$ and $\gamma^{l_0,\ldots,l_s}(j) = (s+j)^{l_0}(j+1)^{l_1} \ldots (j+s)^{l_s}$. We have

$$\begin{aligned} R_\varepsilon^{0^{(k)}\boldsymbol{n}} &= (C_\varepsilon^\bullet \times V_\varepsilon^\bullet)^{0^{(k)}\boldsymbol{n}} \\ &= \sum_{j=0}^{k} \frac{1}{(k-j)!\varepsilon^{k-j}} V_\varepsilon^{0^{(j)}\boldsymbol{n}} \\ &= \frac{(-1)^s x^{\|\boldsymbol{n}\|}}{\varepsilon^k} \sum_{j=0}^{k} \sum_{\substack{l_t \geq 0 \\ 0 \leq t \leq s}} \frac{\varepsilon^{\|\boldsymbol{l}\|}(-1)^{\|\boldsymbol{l}\|-l_0}}{l_0! p_1^{l_1+1} \ldots p_s^{l_s+1}} (\log x)^{l_0} \sum_{j=0}^{k} \frac{(-1)^j \gamma^{l_0,\ldots,l_s}(j)}{j!(k-j)!}. \end{aligned}$$

In order to prove that $R_\varepsilon^{0^{(k)}\boldsymbol{n}}$ is in $\mathcal{A}_+$, it is sufficient to check that, for $\|\boldsymbol{l}\| = l_0 + \cdots + l_s < k$ ($k \geq 1$),

$$\theta^{l_0,\dots,l_s}(k) = \sum_{j=0}^{k} \frac{(-1)^j}{(k-j)!j!}\gamma^{l_0,\dots,l_s}(j) = 0.$$

Let $t_0, \dots, t_s$ be $s+1$ variables, then

$$f_j(t_0,\dots,t_s) = \sum_{\substack{l_t \geq 0\\ 0\leq t\leq s}} \frac{\gamma^{l_0,\dots,l_s}(j)}{l_0!\dots l_s!} t_0^{l_0}\dots t_s^{l_s} = e^{(s+j)t_0+(j+1)t_1+\cdots+(j+s)t_s}.$$

We have

$$\begin{aligned} g_k(t_0,\dots,t_s) &= \sum_{\substack{l_t \geq 0\\ 0\leq t\leq s}} \frac{\theta^{l_0,\dots,l_s}(k)}{l_0!\dots l_s!} t_0^{l_0}\dots t_s^{l_s} \\ &= \sum_{j=0}^{k} \frac{(-1)^j}{(k-j)!j!} e^{j(t_0+\cdots+t_s)} e^{st_0+t_1+2t_2+\cdots+st_s} \\ &= \frac{1}{k!}(1-e^{t_0+\cdots+t_s})^k e^{st_0+t_1+2t_2+\cdots+st_s}. \end{aligned}$$

It becomes clear that, in this series, if $l_0+\cdots+l_s<k$, then $\theta^{l_0,\dots,l_s}(k)=0$ and this proves that $R_\varepsilon^{0^{(k)}\boldsymbol{n}}$ is regular in ε.

In the series defining $R_\varepsilon^{0^{(k)}\boldsymbol{n}}$, the value of $R_0^{0^{(k)}\boldsymbol{n}}$ is then given by

$$\begin{aligned} R_0^{0^{(k)}\boldsymbol{n}} &= \sum_{j=0}^{k} \frac{(-1)^{j+s}x^{\|\boldsymbol{n}\|}}{(k-j)!j!} \sum_{\substack{l_t\geq 0\\ 0\leq t\leq s\\ l_0+\cdots+l_s=k}} \frac{(-1)^{\|\boldsymbol{l}\|-l_0}}{l_0!p_1^{l_1+1}\dots p_s^{l_s+1}}(\log x)^{l_0}\gamma^{l_0,\dots,l_s}(j) \\ &= (-1)^s x^{\|\boldsymbol{n}\|} \sum_{\substack{l_t\geq 0\\ 0\leq t\leq s\\ l_0+\cdots+l_s=k}} \frac{(-1)^{\|\boldsymbol{l}\|-l_0}}{l_0!p_1^{l_1+1}\dots p_s^{l_s+1}}(\log x)^{l_0}\theta^{l_0,\dots,l_s}(k) \\ &= \sum_{j=0}^{k} L^{0^{(k-j)}}.\Big(\sum_{\substack{l_t\geq 0\\ 1\leq t\leq s\\ l_1+\cdots+l_s=j}} \frac{(-1)^{k+s}x^{\|\boldsymbol{n}\|}}{p_1^{l_1+1}\dots p_s^{l_s+1}}\theta^{k-j,l_1,\dots,l_s}(k)\Big). \end{aligned}$$

To prove the second factorization, it remains to prove that

$$\sum_{\substack{l_t\geq 0\\ 1\leq t\leq s\\ l_1+\cdots+l_s=j}} \frac{(-1)^k}{p_1^{l_1+1}\dots p_s^{l_s+1}}\theta^{k-j,l_1,\dots,l_s}(k)$$

does not depend on $k \geq j$. If $l_0 = k - j \geq 0$ and $l_1 + \cdots + l_s = j$, then

$$\begin{aligned}\theta^{k-j,l_1,\ldots,l_s}(k) &= \theta^{l_0,l_1,\ldots,l_s}(l_0 + l_1 + \cdots + l_s)\\ &= \frac{\partial^{l_0+\cdots+l_s}}{\partial t_0^{l_0} \ldots \partial t_s^{l_s}}(g_k(t_0, \ldots, t_s))|_{t_0=\cdots=t_s=0}\end{aligned}$$

but, because of the valuation of $g_k(t_0, \ldots, t_s)$, it is clear that if $l_0 + \cdots + l_s = k$,

$$\begin{aligned}\theta^{k-j,l_1,\ldots,l_s}(k) &= \frac{\partial^{l_0+\cdots+l_s}}{\partial t_0^{l_0} \ldots \partial t_s^{l_s}}\left(\frac{(1 - e^{t_0+\cdots+t_s})^{l_0+\cdots+l_s}}{(l_0 + \cdots + l_s)!}\right)_{t_0=\cdots=t_s=0}\\ &= (-1)^k.\end{aligned}$$

This proves that $S^\bullet$ is well-defined and that, if $\boldsymbol{n} \neq \emptyset$ and $n_1 \neq 0$,

$$S^{0^{(k)}\boldsymbol{n}} = (-1)^s x^{n_1+\cdots+n_s} \sum_{\substack{l_t \geq 0\\ 1\leq t\leq s\\ l_1+\cdots+l_s=k}} \frac{1}{p_1^{l_1+1} \ldots p_s^{l_s+1}}$$

if $p_i = n_1 + \cdots + n_i$. Of course, as $R_0^\bullet$ and $L^\bullet$ are symmetral, it automatically ensures that $S^\bullet$ is symmetral.

7 Conclusion

Our results illustrate, in a simple situation, the interactions between Écalle's work and Hopf algebras and renormalization. The same ideas can be adapted to a wide range of problems of conjugacy of local objects (formal or analytic differential equations, vector fields, difference equations, diffeomorphisms ...). See [6] for details.

At the formal level, one tries to conjugate such objects to a more simple one, for instance their linear part. For differential equations and vector fields, the attempted conjugating map is given by a character on some shuffle Hopf algebra and, when such obstructions as resonance occur, this leads to a ill-defined character. As in our example, some renormalization scheme can be applied and gives interesting results on the existence of "normal forms" and "ramified" conjugating maps. The same holds for difference equations and diffeomorphisms except that the conjugating map is associated to a character on a quasishuffle Hopf algebra (see [6], [8]), that is to say a "symmetrel" mould. In the case of difference equations, such characters are closely related to multizeta values.

In addition to the difficulties of such formal problems, one can also look at the analytic case, that is to say analytic conjugacy of analytic objects. In this case, two new difficulties arise and interacts with renormalization.

1. Mould–comould expansions are not well suited for analytics estimates since many terms contribute to a same monomial in the power series of the conjugating

map. In many cases, a solution can be found in Écalle's work: arborification-coarborification. Roughly speaking, the mould–comould expansion can be reorganized as a series of operators indexed by trees. This often gives better estimates that lead to analyticity and points out a new interaction between Écalle'work and Hopf algebras since such "tree" expansion are closely related to characters on the Connes–Kreimer Hopf algebra of (eventually decorated) trees.

2. The second difficulty comes from the fact that, even after arborification-coarborification, the attempted conjugating map may remain formal but with some Gevrey estimates on the coefficients. One can then obtain "sectorial" analytic diffeomorphisms, using the usual tools of resummation, and this gives rise to a wide range of mathematical problems on the interactions between renormalization and resummation.

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Feynman integrals and multiple polylogarithms

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Abstract. In this article I review the connections between Feynman integrals and multiple polylogarithms. After an introductory section on loop integrals I discuss the Mellin–Barnes transformation and shuffle algebras. In a subsequent section multiple polylogarithms are introduced. Finally, I discuss how certain Feynman integrals evaluate to multiple polylogarithms.

1 Introduction

In this talk I will discuss techniques for the computation of loop integrals, which occur in perturbative calculations in quantum field theory. Particle physics has become a field where precision measurements have become possible. Of course, the increase in experimental precision has to be matched with more accurate calculations from the theoretical side. This is the "raison d'être" for loop calculations: A higher accuracy is reached by including more terms in the perturbative expansion. The complexity of a calculation increases obviously with the number of loops, but also with the number of external particles or the number of non-zero internal masses associated to propagators. To give an idea of the state of the art, specific quantities which are just pure numbers have been computed up to an impressive fourth or third order. Examples are the calculation of the 4-loop contribution to the QCD β-function [1], the calculation of the anomalous magnetic moment of the electron up to three loops [2], and the calculation of the ratio of the total cross section for hadron production to the total cross section for the production of a $\mu^+\mu^-$ pair in electron-positron annihilation to order $O\left(\alpha_s^3\right)$ [3]. Quantities which depend on a single variable are known at the best to the third order. Outstanding examples are the computation of the three-loop Altarelli–Parisi splitting functions [4], [5] or the calculation of the two-loop amplitudes for the most interesting $2 \to 2$ processes [6]–[16]. For the calculation of these amplitudes, the knowledge of certain highly non-trivial two-loop integrals has been essential [17], [18], [19]. The complexity of a two-loop computation increases, if the result depends on more than one variable. An example for a two-loop calculation whose result depends on two variables is the computation of the two-loop amplitudes for $e^+e^- \to 3$ jets [20], [21], [22]. But in general, if more than one variable is involved, we have to content ourselves with next-to-leading order calculations. An example for the state of the art is here the computation of the electro-weak corrections to the process $e^+e^- \to 4$ fermions [23], [24].

From a mathematical point of view loop calculations reveal interesting algebraic structures. Multiple polylogarithms play an important role to express the results of loop calculations. The mathematical aspects will be discussed in this talk. Additional material related to loop calculations can found in the reviews [25]–[28] and the book [29].

This paper is organised as follows: In the next section I review basic facts about Feynman integrals. Section 3 is devoted to the Mellin–Barnes transformation. In Section 4 algebraic structures like shuffle algebras are introduced. Section 5 deals with multiple polylogarithms. Section 6 combines the various aspects and shows, how certain Feynman integrals evaluate to multiple polylogarithms. Finally, Section 7 contains a summary.

2 Feynman integrals

To set the scene let us consider a scalar Feynman graph G. Figure 1 shows an example. In this example there are three external lines and six internal lines. The momenta flowing in or out through the external lines are labelled p_1, p_2 and p_3 and can be taken as fixed vectors. They are constrained by momentum conservation: If all momenta are taken to flow outwards, momentum conservation requires that

$$p_1 + p_2 + p_3 = 0. \tag{2.1}$$

At each vertex of a graph we have again momentum conservation: The sum of all momenta flowing into the vertex equals the sum of all momenta flowing out of the vertex. A graph, where the external momenta determine uniquely all internal momenta is called a tree graph. It can be shown that such a graph does not contain any closed circuit.

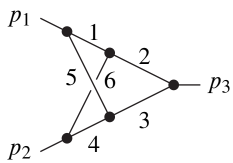

Figure 1. An example of a two-loop Feynman graph with three external legs.

In contrast, graphs which do contain one or more closed circuits are called loop graphs. If we have to specify besides the external momenta in addition l internal momenta in order to determine uniquely all internal momenta we say that the graph contains l loops. In this sense, a tree graph is a graph with zero loops and the graph in Figure 1 contains two loops. Let us agree that we label the l additional internal momenta by k_1 to k_l.

Feynman rules allow us to translate a Feynman graph into a mathematical formula. For a scalar graph we have substitute for each internal line j a propagator

$$\frac{i}{q_j^2 - m_j^2 + i\delta}. \tag{2.2}$$

Here, q_j is the momentum flowing through line j. It is a linear combination of the external momenta p and the loop momenta k:

$$q_j = q_j(p, k). \tag{2.3}$$

m_j is the mass of the particle of line j. The propagator would have a pole for $p_j^2 = m_j^2$, or phrased differently $E_j = \pm\sqrt{\vec{p}_j^{\,2} + m_j^2}$. When integrating over E, the integration contour has to be deformed to avoid these two poles. Causality dictates into which directions the contour has to be deformed. The pole on the negative real axis is avoided by escaping into the lower complex half-plane, the pole at the positive real axis is avoided by a deformation into the upper complex half-plane. Feynman invented the trick to add a small imaginary part $i\delta$ to the denominator, which keeps track of the directions into which the contour has to be deformed. In the following the $i\delta$-term is omitted in order to keep the notation compact.

The Feynman rules tell us also to integrate for each loop over the loop momentum:

$$\int \frac{d^4 k_r}{(2\pi)^4}. \tag{2.4}$$

However, there is a complication: If we proceed naively and write down for each loop an integral over four-dimensional Minkowski space, we end up with ill-defined integrals, since these integrals may contain ultraviolet or infrared divergences! Therefore the first step is to make these integrals well-defined by introducing a regulator. There are several possibilities how this can be done, but the method of dimensional regularisation [30], [31], [32] has almost become a standard, as the calculations in this regularisation scheme turn out to be the simplest. Within dimensional regularisation one replaces the four-dimensional integral over the loop momentum by an D-dimensional integral, where D is now an additional parameter, which can be a non-integer or even a complex number. We consider the result of the integration as a function of D and we are interested in the behaviour of this function as D approaches 4. It is common practice to parameterise the deviation of D from 4 by

$$D = 4 - 2\varepsilon. \tag{2.5}$$

The divergences in loop integrals will manifest themselves in poles in $1/\varepsilon$. In an l-loop integral ultraviolet divergences will lead to poles $1/\varepsilon^l$ at the worst, whereas infrared divergences can lead to poles up to $1/\varepsilon^{2l}$. We will also encounter integrals, where the dimension is shifted by units of two. In these cases we often write

$$D = 2m - 2\varepsilon, \tag{2.6}$$

where m is an integer, and we are again interested in the Laurent series in ε.

Let us now consider a generic scalar l-loop integral I_G in $D = 2m - 2\varepsilon$ dimensions with n propagators, corresponding to a graph G. Let us further make a slight generalisation: For each internal line j the corresponding propagator in the integrand can be raised to a power ν_j. Therefore the integral will depend also on the numbers $\nu_1, \dots, \nu_n$. We define the Feynman integral by

$$I_G = \left(e^{\varepsilon \gamma_E} \mu^{2\varepsilon}\right)^l \int \prod_{r=1}^{l} \frac{d^D k_r}{i \pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}}. \tag{2.7}$$

The momenta q_j of the propagators are linear combinations of the external momenta and the loop momenta. In equation (2.7) there are some overall factors, which I inserted for convenience: μ is an arbitrary mass scale and the factor $\mu^{2\varepsilon}$ ensures that the mass dimension of equation (2.7) is an integer. The factor $e^{\varepsilon \gamma_E}$ avoids a proliferation of Euler's constant

$$\gamma_E = \lim_{n \to \infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \ln n \right) = 0.5772156649\ldots \tag{2.8}$$

in the final result. The integral measure is now $d^D k/(i\pi^{D/2})$ instead of $d^D k/(2\pi)^D$, and each propagator is multiplied by i. The small imaginary parts $i\delta$ in the propagators are not written explicitly.

How to perform the D-dimensional loop integrals ? The first step is to convert the products of propagators into a sum. This can be done with the Feynman parameter technique. In its full generality it is also applicable to cases, where each factor in the denominator is raised to some power ν. The formula reads:

$$\prod_{i=1}^{n} \frac{1}{P_i^{\nu_i}} = \frac{\Gamma(\nu)}{\prod\limits_{i=1}^{n} \Gamma(\nu_i)} \int\limits_0^1 \left(\prod_{i=1}^{n} dx_i \; x_i^{\nu_i - 1} \right) \frac{\delta\left(1 - \sum_{i=1}^{n} x_i\right)}{\left(\sum_{i=1}^{n} x_i P_i\right)^{\nu}}, \qquad \nu = \sum_{i=1}^{n} \nu_i. \tag{2.9}$$

Applied to equation (2.7) we have

$$\sum_{i=1}^{n} x_i P_i = \sum_{i=1}^{n} x_i (-q_i^2 + m_i^2). \tag{2.10}$$

One can now use translational invariance of the D-dimensional loop integrals and shift each loop momentum k_r to complete the square, such that the integrand depends only on k_r^2. Then all D-dimensional loop integrals can be performed. As the integrals over the Feynman parameters still remain, this allows us to treat the D-dimensional loop integrals for Feynman parameter integrals. One arrives at the following Feynman parameter integral [33]:

$$I_G = \left(e^{\varepsilon \gamma_E} \mu^{2\varepsilon}\right)^l \frac{\Gamma(\nu - lD/2)}{\prod\limits_{j=1}^{n} \Gamma(\nu_j)} \int\limits_0^1 \Big(\prod_{j=1}^{n} dx_j \, x_j^{\nu_j - 1} \Big) \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{\nu-(l+1)D/2}}{\mathcal{F}^{\nu - lD/2}}. \tag{2.11}$$

The functions $\mathcal{U}$ and $\mathcal{F}$ depend on the Feynman parameters. If one expresses

$$\sum_{j=1}^{n} x_j (-q_j^2 + m_j^2) = -\sum_{r=1}^{l} \sum_{s=1}^{l} k_r M_{rs} k_s + \sum_{r=1}^{l} 2k_r \cdot Q_r - J, \tag{2.12}$$

where M is a $l \times l$ matrix with scalar entries and Q is a l-vector with fourvectors as entries, one obtains

$$\mathcal{U} = \det(M), \quad \mathcal{F} = \det(M)(-J + QM^{-1}Q). \tag{2.13}$$

Alternatively, the functions $\mathcal{U}$ and $\mathcal{F}$ can be derived from the topology of the corresponding Feynman graph G. Cutting l lines of a given connected l-loop graph such that it becomes a connected tree graph T defines a chord $\mathcal{C}(T, G)$ as being the set of lines not belonging to this tree. The Feynman parameters associated with each chord define a monomial of degree l. The set of all such trees (or 1-trees) is denoted by $\mathcal{T}_1$. The 1-trees $T \in \mathcal{T}_1$ define $\mathcal{U}$ as being the sum over all monomials corresponding to the chords $\mathcal{C}(T, G)$. Cutting one more line of a 1-tree leads to two disconnected trees (T_1, T_2), or a 2-tree. $\mathcal{T}_2$ is the set of all such pairs. The corresponding chords define monomials of degree $l + 1$. Each 2-tree of a graph corresponds to a cut defined by cutting the lines which connected the two now disconnected trees in the original graph. The square of the sum of momenta through the cut lines of one of the two disconnected trees T_1 or T_2 defines a Lorentz invariant

$$s_T = \Big(\sum_{j \in \mathcal{C}(T,G)} p_j \Big)^2. \tag{2.14}$$

The function $\mathcal{F}_0$ is the sum over all such monomials times minus the corresponding invariant. The function $\mathcal{F}$ is then given by $\mathcal{F}_0$ plus an additional piece involving the internal masses m_j. In summary, the functions $\mathcal{U}$ and $\mathcal{F}$ are obtained from the graph as follows:

$$\begin{aligned}
\mathcal{U} &= \sum_{T \in \mathcal{T}_1} \Big[\prod_{j \in \mathcal{C}(T,G)} x_j \Big], \\
\mathcal{F}_0 &= \sum_{(T_1,T_2) \in \mathcal{T}_2} \Big[\prod_{j \in \mathcal{C}(T_1,G)} x_j \Big] (-s_{T_1}), \\
\mathcal{F} &= \mathcal{F}_0 + \mathcal{U} \sum_{j=1}^{n} x_j m_j^2.
\end{aligned} \tag{2.15}$$

In general, $\mathcal{U}$ is a positive semi-definite function. Its vanishing is related to the UV sub-divergences of the graph. Overall UV divergences, if present, will always be contained in the prefactor $\Gamma(\nu - lD/2)$. In the Euclidean region, $\mathcal{F}$ is also a positive semi-definite function of the Feynman parameters x_j.

As an example we consider the graph in Figure 1. For simplicity we assume that all internal propagators are massless. Then the functions $\mathcal{U}$ and $\mathcal{F}$ read:

$$\begin{aligned}
\mathcal{U} &= x_{15}x_{23} + x_{15}x_{46} + x_{23}x_{46}, \\
\mathcal{F} &= (x_1x_3x_4 + x_5x_2x_6 + x_1x_5x_{2346})\left(-p_1^2\right) \\
&\quad + (x_6x_3x_5 + x_4x_1x_2 + x_4x_6x_{1235})\left(-p_2^2\right) \\
&\quad + (x_2x_4x_5 + x_3x_1x_6 + x_2x_3x_{1456})\left(-p_3^2\right).
\end{aligned} \tag{2.16}$$

Here we used the notation that $x_{ij\dots r} = x_i + x_j + \dots + x_r$.

Finally let us remark, that in equation (2.7) we restricted ourselves to scalar integrals, where the numerator of the integrand is independent of the loop momentum. A priori more complicated cases, where the loop momentum appears in the numerator might occur. However, there is a general reduction algorithm, which reduces these tensor integrals to scalar integrals [34], [35]. The price we have to pay is that these scalar integrals involve higher powers of the propagators and/or have shifted dimensions. Therefore we considered in equation (2.6) shifted dimensions and in equation (2.7) arbitrary powers of the propagators. In conclusion, the integrals of the form as in equation (2.7) are the most general loop integrals we have to solve.

3 The Mellin–Barnes transformation

In Section 2 we saw that the Feynman parameter integrals depend on two graph polynomials $\mathcal{U}$ and $\mathcal{F}$, which are homogeneous functions of the Feynman parameters. In this section we will continue the discussion how these integrals can be performed and exchanged against a (multiple) sum over residues. The case, where the two polynomials are absent is particular simple:

$$\int_0^1 \left(\prod_{j=1}^{n} dx_j\, x_j^{\nu_j-1}\right)\delta(1-\sum_{i=1}^{n} x_i) = \frac{\prod_{j=1}^{n}\Gamma(\nu_j)}{\Gamma(\nu_1+\dots+\nu_n)}. \tag{3.1}$$

With the help of the Mellin–Barnes transformation we now reduce the general case to equation (3.1). The Mellin–Barnes transformation reads

$$\begin{aligned}
(A_1 + A_2 + \dots + A_n)^{-c} &= \frac{1}{\Gamma(c)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} d\sigma_1 \dots \int_{-i\infty}^{i\infty} d\sigma_{n-1} \\
&\times \Gamma(-\sigma_1)\dots\Gamma(-\sigma_{n-1})\Gamma(\sigma_1+\dots+\sigma_{n-1}+c)\; A_1^{\sigma_1} \dots A_{n-1}^{\sigma_{n-1}} A_n^{-\sigma_1-\dots-\sigma_{n-1}-c}.
\end{aligned} \tag{3.2}$$

Each contour is such that the poles of $\Gamma(-\sigma)$ are to the right and the poles of $\Gamma(\sigma+c)$ are to the left. This transformation can be used to convert the sum of monomials of the polynomials $\mathcal{U}$ and $\mathcal{F}$ into a product, such that all Feynman parameter integrals are of the form of equation (3.1). As this transformation converts sums into products it is the "inverse" of Feynman parametrisation. Equation (3.2) is derived from the theory of Mellin transformations: Let $h(x)$ be a function which is bounded by a power law for $x \to 0$ and $x \to \infty$, e.g.

$$\begin{aligned} |h(x)| &\le K x^{-c_0} && \text{for } x \to 0, \\ |h(x)| &\le K' x^{c_1} && \text{for } x \to \infty. \end{aligned} \tag{3.3}$$

Then the Mellin transform is defined for $c_0 < \operatorname{Re}\sigma < c_1$ by

$$h_M(\sigma) = \int\limits_0^\infty dx\, h(x)\, x^{\sigma-1}. \tag{3.4}$$

The inverse Mellin transform is given by

$$h(x) = \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} d\sigma\, h_M(\sigma)\, x^{-\sigma}. \tag{3.5}$$

The integration contour is parallel to the imaginary axis and $c_0 < \operatorname{Re}\gamma < c_1$. As an example for the Mellin transform we consider the function

$$h(x) = \frac{x^c}{(1+x)^c} \tag{3.6}$$

with Mellin transform $h_M(\sigma) = \Gamma(-\sigma)\Gamma(\sigma+c)/\Gamma(c)$. For $\operatorname{Re}(-c) < \operatorname{Re}\gamma < 0$ we have

$$\frac{x^c}{(1+x)^c} = \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} d\sigma\, \frac{\Gamma(-\sigma)\Gamma(\sigma+c)}{\Gamma(c)}\, x^{-\sigma}. \tag{3.7}$$

From equation (3.7) one obtains with $x = B/A$ the Mellin–Barnes formula

$$(A+B)^{-c} = \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} d\sigma\, \frac{\Gamma(-\sigma)\Gamma(\sigma+c)}{\Gamma(c)}\, A^\sigma B^{-\sigma-c}. \tag{3.8}$$

Equation (3.2) is then obtained by repeated use of equation (3.8).

With the help of equation (3.1) and equation (3.2) we may exchange the Feynman parameter integrals against multiple contour integrals. A single contour integral is of the form

$$I = \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} d\sigma \frac{\Gamma(\sigma + a_1) \dots \Gamma(\sigma + a_m)}{\Gamma(\sigma + c_2) \dots \Gamma(\sigma + c_p)} \frac{\Gamma(-\sigma + b_1) \dots \Gamma(-\sigma + b_n)}{\Gamma(-\sigma + d_1) \dots \Gamma(-\sigma + d_q)} x^{-\sigma}. \tag{3.9}$$

If $\max(\mathrm{Re}(-a_1), \dots, \mathrm{Re}(-a_m)) < \min(\mathrm{Re}(b_1), \dots, \mathrm{Re}(b_n))$ the contour can be chosen as a straight line parallel to the imaginary axis with

$$\max(\mathrm{Re}(-a_1), \dots, \mathrm{Re}(-a_m)) < \mathrm{Re}\,\gamma < \min(\mathrm{Re}(b_1), \dots, \mathrm{Re}(b_n)), \tag{3.10}$$

otherwise the contour is indented, such that the residues of $\Gamma(\sigma + a_1)$, ..., $\Gamma(\sigma + a_m)$ are to the right of the contour, whereas the residues of $\Gamma(-\sigma + b_1)$, ..., $\Gamma(-\sigma + b_n)$ are to the left of the contour. We further set

$$\begin{aligned}
\alpha &= m + n - p - q, \\
\beta &= m - n - p + q, \\
\lambda &= \mathrm{Re}\left(\sum_{j=1}^{m} a_j + \sum_{j=1}^{n} b_j - \sum_{j=1}^{p} c_j - \sum_{j=1}^{q} d_j\right) - \frac{1}{2}(m + n - p - q).
\end{aligned} \tag{3.11}$$

Then the integral equation (3.9) converges absolutely for $\alpha > 0$ [36] and defines an analytic function in

$$|\arg x| < \min\left(\pi, \alpha\frac{\pi}{2}\right). \tag{3.12}$$

The integral equation (3.9) is most conveniently evaluated with the help of the residuum theorem by closing the contour to the left or to the right. Therefore we need to know under which conditions the semi-circle at infinity used to close the contour gives a vanishing contribution. This is obviously the case for $|x| < 1$ if we close the contour to the left, and for $|x| > 1$, if we close the contour to the right. The case $|x| = 1$ deserves some special attention. One can show that in the case $\beta = 0$ the semi-circle gives a vanishing contribution, provided

$$\lambda < -1. \tag{3.13}$$

To sum up all residues which lie inside the contour it is useful to know the residues of the Gamma function:

$$\begin{aligned}
\mathrm{res}\,(\Gamma(\sigma + a), \sigma = -a - n) &= \frac{(-1)^n}{n!}, \\
\mathrm{res}\,(\Gamma(-\sigma + a), \sigma = a + n) &= -\frac{(-1)^n}{n!}.
\end{aligned} \tag{3.14}$$

In general, one obtains (multiple) sum over residues. In particular simple cases the contour integrals can be performed in closed form with the help of two lemmas of

Barnes. Barnes first lemma states that

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} d\sigma\ \Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c-\sigma)\Gamma(d-\sigma) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \tag{3.15}$$

if none of the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ coincides with the ones from $\Gamma(c-\sigma)\Gamma(d-\sigma)$. Barnes second lemma reads

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} d\sigma\ \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c+\sigma)\Gamma(d-\sigma)\Gamma(e-\sigma)}{\Gamma(a+b+c+d+e+\sigma)} = \frac{\Gamma(a+d)\Gamma(b+d)\Gamma(c+d)\Gamma(a+e)\Gamma(b+e)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)}. \tag{3.16}$$

Although the Mellin–Barnes transformation has been known for a long time, the method has seen a revival in applications in recent years [37], [38], [39], [17], [18], [19], [40]–[49].

4 Shuffle algebras

Before we continue the discussion of loop integrals, it is useful to discuss first shuffle algebras and generalisations thereof from an algebraic viewpoint. Consider a set of letters A. The set A is called the alphabet. A word is an ordered sequence of letters:

$$w = l_1 l_2 \dots l_k. \tag{4.1}$$

The word of length zero is denoted by e. Let K be a field and consider the vector space of words over K. A shuffle algebra A on the vector space of words is defined by

$$(l_1 l_2 \dots l_k)\cdot(l_{k+1}\dots l_r) = \sum_{\text{shuffles } \sigma} l_{\sigma(1)} l_{\sigma(2)} \dots l_{\sigma(r)}, \tag{4.2}$$

where the sum runs over all permutations σ, which preserve the relative order of $1, 2, \dots, k$ and of $k+1, \dots, r$. The name "shuffle algebra" is related to the analogy of shuffling cards: If a deck of cards is split into two parts and then shuffled, the relative order within the two individual parts is conserved. The empty word e is the unit in this algebra:

$$e\cdot w = w\cdot e = w. \tag{4.3}$$

A recursive definition of the shuffle product is given by

$$(l_1 l_2 \dots l_k)\cdot(l_{k+1}\dots l_r) = l_1\left[(l_2\dots l_k)\cdot(l_{k+1}\dots l_r)\right] + l_{k+1}\left[(l_1 l_2\dots l_k)\cdot(l_{k+2}\dots l_r)\right] \tag{4.4}$$

It is well known fact that the shuffle algebra is actually a (non-cocommutative) Hopf algebra [50]. In this context let us briefly review the definitions of a coalgebra, a bialgebra and a Hopf algebra, which are closely related: First note that the unit in an algebra can be viewed as a map from K to A and that the multiplication can be viewed as a map from the tensor product $A \otimes A$ to A (e.g. one takes two elements from A, multiplies them and gets one element out).

A coalgebra has instead of multiplication and unit the dual structures: a comultiplication Δ and a counit $\bar{e}$. The counit is a map from A to K, whereas comultiplication is a map from A to $A \otimes A$. Note that comultiplication and counit go in the reverse direction compared to multiplication and unit. We will always assume that the comultiplication is coassociative. The general form of the coproduct is

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}, \tag{4.5}$$

where $a_i^{(1)}$ denotes an element of A appearing in the first slot of $A \otimes A$ and $a_i^{(2)}$ correspondingly denotes an element of A appearing in the second slot. Sweedler's notation [51] consists in dropping the dummy index i and the summation symbol:

$$\Delta(a) = a^{(1)} \otimes a^{(2)} \tag{4.6}$$

The sum is implicitly understood. This is similar to Einstein's summation convention, except that the dummy summation index i is also dropped. The superscripts $^{(1)}$ and $^{(2)}$ indicate that a sum is involved.

A bialgebra is an algebra and a coalgebra at the same time, such that the two structures are compatible with each other. Using Sweedler's notation, the compatibility between the multiplication and comultiplication is expressed as

$$\Delta\left(a \cdot b\right) = \left(a^{(1)} \cdot b^{(1)}\right) \otimes \left(a^{(2)} \cdot b^{(2)}\right). \tag{4.7}$$

A Hopf algebra is a bialgebra with an additional map from A to A, called the antipode S, which fulfils

$$a^{(1)} \cdot S\left(a^{(2)}\right) = S\left(a^{(1)}\right) \cdot a^{(2)} = 0 \quad \text{for } a \neq e. \tag{4.8}$$

With this background at hand we can now state the coproduct, the counit and the antipode for the shuffle algebra: The counit $\bar{e}$ is given by:

$$\bar{e}\left(e\right) = 1, \quad \bar{e}\left(l_1 l_2 \dots l_n\right) = 0. \tag{4.9}$$

The coproduct Δ is given by:

$$\Delta\left(l_1 l_2 \dots l_k\right) = \sum_{j=0}^{k} \left(l_{j+1} \dots l_k\right) \otimes \left(l_1 \dots l_j\right). \tag{4.10}$$

The antipode S is given by:

$$S\left(l_1 l_2 \dots l_k\right) = (-1)^k \, l_k l_{k-1} \dots l_2 l_1. \tag{4.11}$$

The shuffle algebra is generated by the Lyndon words. If one introduces a lexicographic ordering on the letters of the alphabet A, a Lyndon word is defined by the property

$$w < v \tag{4.12}$$

for any sub-words u and v such that $w = uv$.

An important example for a shuffle algebra are iterated integrals. Let $[a, b]$ be a segment of the real line and $f_1,\ f_2,\ \ldots$functions on this interval. Let us define the following iterated integrals:

$$I(f_1, f_2, \ldots, f_k; a, b) = \int\limits_a^b f_1(t_1) dt_1 \int\limits_a^{t_1} f_2(t_2) dt_2 \ldots \int\limits_a^{t_{k-1}} f_k(t_k) dt_k \tag{4.13}$$

For fixed a and b we have a shuffle algebra:

$$\begin{aligned} I(f_1, f_2, \ldots, f_k; a, b) \cdot I(f_{k+1}, \ldots, f_r; a, b) \\ = \sum_{\text{shuffles } \sigma} I(f_{\sigma(1)}, f_{\sigma(2)}, \ldots, f_{\sigma(r)}; a, b), \end{aligned} \tag{4.14}$$

where the sum runs over all permutations σ, which preserve the relative order of $1, 2, \ldots, k$ and of $k+1, \ldots, r$. The proof is sketched in Figure 2. The two outermost integrations are recursively replaced by integrations over the upper and lower triangle.

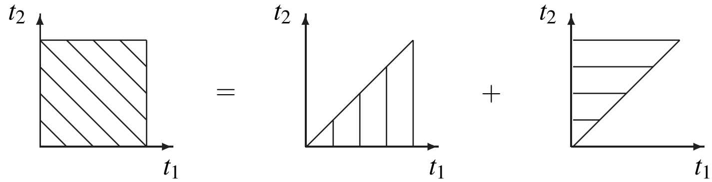

Figure 2. Sketch of the proof for the shuffle product of two iterated integrals. The integral over the square is replaced by two integrals over the upper and lower triangle.

We now consider generalisations of shuffle algebras. Assume that for the set of letters we have an additional operation

$$\begin{aligned} (.,.)\colon A \otimes A &\to A, \\ l_1 \otimes l_2 &\mapsto (l_1, l_2), \end{aligned} \tag{4.15}$$

which is commutative and associative. Then we can define a new product of words recursively through

$$\begin{aligned} (l_1 l_2 \ldots l_k) * (l_{k+1} \ldots l_r) = l_1 \left[(l_2 \ldots l_k) * (l_{k+1} \ldots l_r)\right] \\ + l_{k+1} \left[(l_1 l_2 \ldots l_k) * (l_{k+2} \ldots l_r)\right] \\ + (l_1, l_{k+1}) \left[(l_2 \ldots l_k) * (l_{k+2} \ldots l_r)\right]. \end{aligned} \tag{4.16}$$

This product is a generalisation of the shuffle product and differs from the recursive definition of the shuffle product in equation (4.4) through the extra term in the last line. This modified product is known under the names quasi-shuffle product [52], mixable shuffle product [53] or stuffle product [54]. Quasi-shuffle algebras are Hopf algebras. Comultiplication and counit are defined as for the shuffle algebras. The counit $\bar{e}$ is given by

$$\bar{e}\,(e) = 1, \quad \bar{e}\,(l_1 l_2 \dots l_n) = 0. \tag{4.17}$$

The coproduct Δ is given by:

$$\Delta\,(l_1 l_2 \dots l_k) = \sum_{j=0}^{k} \left(l_{j+1} \dots l_k\right) \otimes \left(l_1 \dots l_j\right). \tag{4.18}$$

The antipode S is recursively defined through

$$S\,(l_1 l_2 \dots l_k) = -l_1 l_2 \dots l_k - \sum_{j=1}^{k-1} S\left(l_{j+1} \dots l_k\right) * \left(l_1 \dots l_j\right). \tag{4.19}$$

An example for a quasi-shuffle algebra are nested sums. Let n_a and n_b be integers with $n_a < n_b$ and let f_1, f_2, ...be functions defined on the integers. We consider the following nested sums:

$$S(f_1, f_2, \dots, f_k; n_a, n_b) = \sum_{i_1=n_a}^{n_b} f_1(i_1) \sum_{i_2=n_a}^{i_1-1} f_2(i_2) \cdots \sum_{i_k=n_a}^{i_{k-1}-1} f_k(i_k) \tag{4.20}$$

For fixed n_a and n_b we have a quasi-shuffle algebra:

$$\begin{aligned} & S(f_1, f_2, \dots, f_k; n_a, n_b) * S(f_{k+1}, \dots, f_r; n_a, n_b) \\ & = \sum_{i_1=n_a}^{n_b} f_1(i_1)\; S(f_2, \dots, f_k; n_a, i_1 - 1) * S(f_{k+1}, \dots, f_r; n_a, i_1 - 1) \\ & \quad + \sum_{j_1=n_a}^{n_b} f_k(j_1)\; S(f_1, f_2, \dots, f_k; n_a, j_1 - 1) * S(f_{k+2}, \dots, f_r; n_a, j_1 - 1) \\ & \quad + \sum_{i=n_a}^{n_b} f_1(i) f_k(i)\; S(f_2, \dots, f_k; n_a, i - 1) * S(f_{k+2}, \dots, f_r; n_a, i - 1) \end{aligned} \tag{4.21}$$

Note that the product of two letters corresponds to the point-wise product of the two functions:

$$(f_i, f_j)(n) = f_i(n) f_j(n). \tag{4.22}$$

The proof that nested sums obey the quasi-shuffle algebra is sketched in Figure 3. The outermost sums of the nested sums on the l.h.s of (4.21) are split into the three regions indicated in Figure 3.

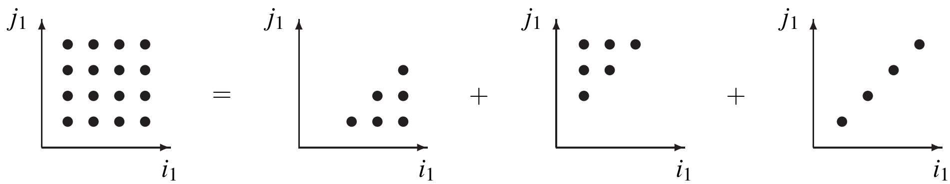

Figure 3. Sketch of the proof for the quasi-shuffle product of nested sums. The sum over the square is replaced by the sum over the three regions on the r.h.s.

5 Multiple polylogarithms

In the previous section we have seen that iterated integrals form a shuffle algebra, while nested sums form a quasi-shuffle algebra. In this context multiple polylogarithms form an interesting class of functions. They have a representation as iterated integrals as well as nested sums. Therefore multiple polylogarithms form a shuffle algebra as well as a quasi-shuffle algebra. The two algebra structures are independent. Let us start with the representation as nested sums. The multiple polylogarithms are defined by

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \sum_{i_1>i_2>\ldots>i_k>0} \frac{x_1^{i_1}}{{i_1}^{m_1}} \cdots \frac{x_k^{i_k}}{{i_k}^{m_k}}. \tag{5.1}$$

The multiple polylogarithms are generalisations of the classical polylogarithms $\mathrm{Li}_n(x)$ [55], whose most prominent examples are

$$\mathrm{Li}_1(x) = \sum_{i_1=1}^{\infty} \frac{x^{i_1}}{i_1} = -\ln(1-x), \quad \mathrm{Li}_2(x) = \sum_{i_1=1}^{\infty} \frac{x^{i_1}}{i_1^2}, \tag{5.2}$$

as well as Nielsen's generalised polylogarithms [56]

$$S_{n,p}(x) = \mathrm{Li}_{n+1,1,\ldots,1}(x, \underbrace{1,\ldots,1}_{p-1}), \tag{5.3}$$

and the harmonic polylogarithms [57]

$$H_{m_1,\ldots,m_k}(x) = \mathrm{Li}_{m_1,\ldots,m_k}(x, \underbrace{1,\ldots,1}_{k-1}). \tag{5.4}$$

Multiple polylogarithms have been studied extensively in the literature by physicists [57]–[70] and mathematicians [54] and [71]–[81].

In addition, multiple polylogarithms have an integral representation. To discuss the integral representation it is convenient to introduce for $z_k \neq 0$ the following functions

$$G(z_1,\ldots,z_k;y) = \int\limits_0^y \frac{dt_1}{t_1-z_1} \int\limits_0^{t_1} \frac{dt_2}{t_2-z_2} \cdots \int\limits_0^{t_{k-1}} \frac{dt_k}{t_k-z_k}. \tag{5.5}$$

In this definition one variable is redundant due to the following scaling relation:

$$G(z_1,\dots,z_k;y) = G(xz_1,\dots,xz_k;xy) \tag{5.6}$$

If one further defines

$$g(z;y) = \frac{1}{y-z}, \tag{5.7}$$

then one has

$$\frac{d}{dy} G(z_1,\dots,z_k;y) = g(z_1;y)G(z_2,\dots,z_k;y) \tag{5.8}$$

and

$$G(z_1,z_2,\dots,z_k;y) = \int\limits_0^y dt\; g(z_1;t)G(z_2,\dots,z_k;t). \tag{5.9}$$

One can slightly enlarge the set and define $G(0,\dots,0;y)$ with k zeros for z_1 to z_k to be

$$G(0,\dots,0;y) = \frac{1}{k!}(\ln y)^k. \tag{5.10}$$

This permits us to allow trailing zeros in the sequence $(z_1,\dots,z_k)$ by defining the function G with trailing zeros via (5.9) and (5.10). To relate the multiple polylogarithms to the functions G it is convenient to introduce the following short-hand notation:

$$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) = G(\underbrace{0,\dots,0}_{m_1-1},z_1,\dots,z_{k-1},\underbrace{0\dots,0}_{m_k-1},z_k;y) \tag{5.11}$$

Here, all z_j for $j=1,\dots,k$ are assumed to be non-zero. One then finds

$$\mathrm{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k) = (-1)^k G_{m_1,\dots,m_k}\left(\frac{1}{x_1},\frac{1}{x_1x_2},\dots,\frac{1}{x_1\dots x_k};1\right). \tag{5.12}$$

The inverse formula reads

$$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) = (-1)^k\,\mathrm{Li}_{m_1,\dots,m_k}\left(\frac{y}{z_1},\frac{z_1}{z_2},\dots,\frac{z_{k-1}}{z_k}\right). \tag{5.13}$$

Equation (5.12) together with (5.11) and (5.5) defines an integral representation for the multiple polylogarithms. To make this more explicit I first introduce some notation for iterated integrals

$$\int\limits_0^\Lambda \frac{dt}{t-a_n}\circ\dots\circ\frac{dt}{t-a_1} = \int\limits_0^\Lambda \frac{dt_n}{t_n-a_n}\int\limits_0^{t_n}\frac{dt_{n-1}}{t_{n-1}-a_{n-1}}\times\dots\times\int\limits_0^{t_2}\frac{dt_1}{t_1-a_1} \tag{5.14}$$

and the short hand notation:

$$\int\limits_0^\Lambda \left(\frac{dt}{t}\circ\right)^m \frac{dt}{t-a} = \int\limits_0^\Lambda \underbrace{\frac{dt}{t}\circ\dots\circ\frac{dt}{t}}_{m\text{times}}\circ\frac{dt}{t-a}. \tag{5.15}$$

The integral representation for $\mathrm{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k)$ reads then

$$\begin{aligned}\mathrm{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k) = (-1)^k \int_0^1 \left(\frac{dt}{t}\circ\right)^{m_1-1} \frac{dt}{t-b_1}\\ \circ \left(\frac{dt}{t}\circ\right)^{m_2-1} \frac{dt}{t-b_2} \circ \dots \circ \left(\frac{dt}{t}\circ\right)^{m_k-1} \frac{dt}{t-b_k},\end{aligned} \tag{5.16}$$

where the b_j's are related to the x_j's

$$b_j = \frac{1}{x_1 x_2 \dots x_j}. \tag{5.17}$$

Up to now we treated multiple polylogarithms from an algebraic point of view. Equally important are the analytical properties, which are needed for an efficient numerical evaluation. As an example I first discuss the numerical evaluation of the dilogarithm [82]:

$$\mathrm{Li}_2(x) = -\int_0^x dt \frac{\ln(1-t)}{t} = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \tag{5.18}$$

The power series expansion can be evaluated numerically, provided $|x| < 1$. Using the functional equations

$$\begin{aligned}\mathrm{Li}_2(x) &= -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2}\left(\ln(-x)\right)^2,\\ \mathrm{Li}_2(x) &= -\mathrm{Li}_2(1-x) + \frac{\pi^2}{6} - \ln(x)\ln(1-x).\end{aligned} \tag{5.19}$$

any argument of the dilogarithm can be mapped into the region $|x| \le 1$ and $-1 \le \mathrm{Re}(x) \le 1/2$. The numerical computation can be accelerated by using an expansion in $[-\ln(1-x)]$ and the Bernoulli numbers B_i:

$$\mathrm{Li}_2(x) = \sum_{i=0}^{\infty} \frac{B_i}{(i+1)!} \left(-\ln(1-x)\right)^{i+1}. \tag{5.20}$$

The generalisation to multiple polylogarithms proceeds along the same lines [67]: Using the integral representation

$$\begin{aligned}&G_{m_1,\dots,m_k}(z_1, z_2, \dots, z_k; y) \\ &= \int_0^y \left(\frac{dt}{t}\circ\right)^{m_1-1} \frac{dt}{t-z_1} \left(\frac{dt}{t}\circ\right)^{m_2-1} \frac{dt}{t-z_2} \dots \left(\frac{dt}{t}\circ\right)^{m_k-1} \frac{dt}{t-z_k}\end{aligned} \tag{5.21}$$

one transforms all arguments into a region, where one has a converging power series expansion:

$$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) = \sum_{j_1=1}^{\infty} \dots \sum_{j_k=1}^{\infty} \frac{1}{\left(j_1+\dots+j_k\right)^{m_1}} \left(\frac{y}{z_1}\right)^{j_1} \times \frac{1}{(j_2+\dots+j_k)^{m_2}} \left(\frac{y}{z_2}\right)^{j_2} \dots \frac{1}{(j_k)^{m_k}} \left(\frac{y}{z_k}\right)^{j_k}. \tag{5.22}$$

The multiple polylogarithms satisfy the Hölder convolution [54]. For $z_1 \neq 1$ and $z_w \neq 0$ this identity reads

$$G(z_1,\dots,z_w;1) = \sum_{j=0}^{w} (-1)^j G\left(1-z_j,1-z_{j-1},\dots,1-z_1;1-\frac{1}{p}\right) G\left(z_{j+1},\dots,z_w;\frac{1}{p}\right). \tag{5.23}$$

The Hölder convolution can be used to accelerate the convergence for the series representation of the multiple polylogarithms.

6 Laurent expansion of Feynman integrals

Let us return to the question on how to compute Feynman integrals. In Section 3 we saw how to obtain from the Mellin–Barnes transformation (multiple) sums by closing the integration contours and summing up the residues. As a simple example let us consider that the sum of residues is equal to

$$\sum_{i=0}^{\infty} \frac{\Gamma(i+a_1+t_1\varepsilon)\Gamma(i+a_2+t_2\varepsilon)}{\Gamma(i+1)\Gamma(i+a_3+t_3\varepsilon)} x^i \tag{6.1}$$

Here a_1, a_2 and a_3 are assumed to be integers. Up to prefactors the expression in equation (6.1) is a hyper-geometric function ${}_2F_1$. We are interested in the Laurent expansion of this expression in the small parameter ε. The basic formula for the expansion of Gamma functions reads

$$\Gamma(n+\varepsilon) = \Gamma(1+\varepsilon)\Gamma(n)\left[1+\varepsilon Z_1(n-1)+\varepsilon^2 Z_{11}(n-1) + \varepsilon^3 Z_{111}(n-1)+\dots+\varepsilon^{n-1} Z_{11\dots1}(n-1)\,\right], \tag{6.2}$$

where $Z_{m_1,\dots,m_k}(n)$ are Euler–Zagier sums defined by

$$Z_{m_1,\dots,m_k}(n) = \sum_{n\geq i_1>i_2>\dots>i_k>0} \frac{1}{i_1{}^{m_1}} \dots \frac{1}{i_k{}^{m_k}}. \tag{6.3}$$

This motivates the following definition of a special form of nested sums, called Z-sums:

$$Z(n;m_1,\ldots,m_k;x_1,\ldots,x_k) = \sum_{n\geq i_1>i_2>\ldots>i_k>0} \frac{x_1^{i_1}}{i_1{}^{m_1}} \cdots \frac{x_k^{i_k}}{i_k{}^{m_k}}. \tag{6.4}$$

k is called the depth of the Z-sum and $w = m_1 + \cdots + m_k$ is called the weight. If the sums go to infinity ($n = \infty$) the Z-sums are multiple polylogarithms:

$$Z(\infty;m_1,\ldots,m_k;x_1,\ldots,x_k) = \mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k). \tag{6.5}$$

For $x_1 = \cdots = x_k = 1$ the definition reduces to the Euler–Zagier sums [83], [84]:

$$Z(n;m_1,\ldots,m_k;1,\ldots,1) = Z_{m_1,\ldots,m_k}(n). \tag{6.6}$$

For $n = \infty$ and $x_1 = \cdots = x_k = 1$ the sum is a multiple ζ-value [54]:

$$Z(\infty;m_1,\ldots,m_k;1,\ldots,1) = \zeta_{m_1,\ldots,m_k}. \tag{6.7}$$

The usefulness of the Z-sums lies in the fact, that they interpolate between multiple polylogarithms and Euler–Zagier sums. The Z-sums form a quasi-shuffle algebra.

Using $\Gamma(x+1) = x\Gamma(x)$, partial fractioning and an adjustment of the summation index one can transform equation (6.1) into terms of the form

$$\sum_{i=1}^{\infty} \frac{\Gamma(i+t_1\varepsilon)\Gamma(i+t_2\varepsilon)}{\Gamma(i)\Gamma(i+t_3\varepsilon)} \cdot \frac{x^i}{i^m}, \tag{6.8}$$

where m is an integer. Now using equation (6.2) one obtains

$$\Gamma(1+\varepsilon)\sum_{i=1}^{\infty} \frac{(1+\varepsilon t_1 Z_1(i-1)+\cdots)(1+\varepsilon t_2 Z_1(i-1)+\cdots)}{(1+\varepsilon t_3 Z_1(i-1)+\cdots)} \cdot \frac{x^i}{i^m}. \tag{6.9}$$

Inverting the power series in the denominator and truncating in ε one obtains in each order in ε terms of the form

$$\sum_{i=1}^{\infty} \frac{x^i}{i^m} Z_{m_1\ldots m_k}(i-1) Z_{m'_1\ldots m'_l}(i-1) Z_{m''_1\ldots m''_n}(i-1) \tag{6.10}$$

Using the quasi-shuffle product for Z-sums the three Euler–Zagier sums can be reduced to single Euler–Zagier sums and one finally arrives at terms of the form

$$\sum_{i=1}^{\infty} \frac{x^i}{i^m} Z_{m_1\ldots m_k}(i-1), \tag{6.11}$$

which are special cases of multiple polylogarithms, called harmonic polylogarithms $H_{m,m_1,\ldots,m_k}(x)$. This completes the algorithm for the expansion in ε for sums of the form as in equation (6.1).

The Hopf algebra of Z-sums has additional structures if we allow expressions of the form

$$\frac{x_0^n}{n^{m_0}} Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k), \tag{6.12}$$

e.g. Z-sums multiplied by a letter. Then the following convolution product

$$\sum_{i=1}^{n-1} \frac{x^i}{i^m} Z(i-1; \ldots) \frac{y^{n-i}}{(n-i)^{m'}} Z(n-i-1; \ldots) \tag{6.13}$$

can again be expressed in terms of expressions of the form (6.12). In addition there is a conjugation, e.g. sums of the form

$$-\sum_{i=1}^{n} \binom{n}{i} (-1)^i \frac{x^i}{i^m} Z(i; \ldots) \tag{6.14}$$

can also be reduced to terms of the form (6.12). The name conjugation stems from the following fact: To any function $f(n)$ of an integer variable n one can define a conjugated function $C \circ f(n)$ as the following sum

$$C \circ f(n) = \sum_{i=1}^{n} \binom{n}{i} (-1)^i f(i). \tag{6.15}$$

Then conjugation satisfies the following two properties:

$$\begin{aligned} C \circ 1 &= 1, \\ C \circ C \circ f(n) &= f(n). \end{aligned} \tag{6.16}$$

Finally there is the combination of conjugation and convolution, e.g. sums of the form

$$-\sum_{i=1}^{n-1} \binom{n}{i} (-1)^i \frac{x^i}{i^m} Z(i; \ldots) \frac{y^{n-i}}{(n-i)^{m'}} Z(n-i; \ldots) \tag{6.17}$$

can also be reduced to terms of the form (6.12). These properties can be used to expand more complicated transcendental functions like

$$\begin{aligned} &\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\Gamma(i+a_1)}{\Gamma(i+a_1')} \cdots \frac{\Gamma(i+a_k)}{\Gamma(i+a_k')} \frac{\Gamma(j+b_1)}{\Gamma(j+b_1')} \cdots \\ &\qquad \cdots \frac{\Gamma(j+b_l)}{\Gamma(j+b_l')} \frac{\Gamma(i+j+c_1)}{\Gamma(i+j+c_1')} \cdots \frac{\Gamma(i+j+c_m)}{\Gamma(i+j+c_m')} x^i y^j \end{aligned} \tag{6.18}$$

or

$$\begin{aligned} &\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j} \frac{\Gamma(i+a_1)}{\Gamma(i+a_1')} \cdots \frac{\Gamma(i+a_k)}{\Gamma(i+a_k')} \frac{\Gamma(j+b_1)}{\Gamma(j+b_1')} \cdots \\ &\qquad \cdots \frac{\Gamma(j+b_l)}{\Gamma(j+b_l')} \frac{\Gamma(i+j+c_1)}{\Gamma(i+j+c_1')} \cdots \frac{\Gamma(i+j+c_m)}{\Gamma(i+j+c_m')} x^i y^j. \end{aligned} \tag{6.19}$$

Examples for functions of this type are the first and second Appell function F_1 and F_2. Note that in these examples there are always as many Gamma functions in the numerator as in the denominator. We assume that all a_n, a'_n, b_n, b'_n, c_n and c'_n are of the form "integer + const $\cdot \varepsilon$".

The first type can be generalised to the form "rational number + const $\cdot \varepsilon$", if the Gamma functions always occur in ratios of the form

$$\frac{\Gamma(n + a - \frac{p}{q} + b\varepsilon)}{\Gamma(n + c - \frac{p}{q} + d\varepsilon)}, \tag{6.20}$$

where the same rational number p/q occurs in the numerator and in the denominator [66]. In this case we have to replace equation (6.2) by

$$\Gamma\left(n + 1 - \frac{p}{q} + \varepsilon\right) = \frac{\Gamma\left(1 - \frac{p}{q} + \varepsilon\right)\Gamma\left(n + 1 - \frac{p}{q}\right)}{\Gamma\left(1 - \frac{p}{q}\right)} \tag{6.21}$$
$$\times \exp\left(-\frac{1}{q}\sum_{l=0}^{q-1}\left(r_q^l\right)^p \sum_{k=1}^{\infty} \varepsilon^k \frac{(-q)^k}{k} Z(q \cdot n; k; r_q^l)\right),$$

which introduces the q-th roots of unity

$$r_q^p = \exp\left(\frac{2\pi i p}{q}\right). \tag{6.22}$$

In summary these techniques allow a systematic procedure for the computation of Feynman integrals, if certain conditions are met. These conditions require that factors of Gamma functions are balanced like in equation (6.18) or equation (6.19) [63], [66]. The algebraic properties of nested sums and iterated integrals discussed here are well-suited for an implementation into a computer algebra system and several packages for these manipulations exist [58], [85], [86], [87], [88].

7 Conclusions

In this article I discussed the mathematical structures underlying the computation of Feynman loop integrals. One encounters iterated structures as nested sums or iterated integrals, which form a Hopf algebra with a shuffle or quasi-shuffle product. Of particular importance are multiple polylogarithms. The algebraic properties of these functions are very rich: They form at the same time a shuffle algebra as well as a quasi-shuffle algebra. Based on these algebraic structures I discussed algorithms which evaluate Feynman integrals to multiple polylogarithms.

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